DOUBLY STOCHASTIC POISSON PROCESS AND THE PRICING OF CATASTROPHE REINSURANCE CONTRACT

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ABSTRACT

We use a doubly stochastic Poisson process (or the Cox process) to model the claim arrival process for catastrophic events. The shot noise process is used for the claim intensity function within the Cox process. The Cox process with shot noise intensity is examined by piecewise deterministic Markov process theory. We apply the Cox process incorporating the shot noise process as its intensity to price a stop-loss catastrophe reinsurance contract. The asymptotic (stationary) distribution of the claim intensity is used to derive pricing formulae for a stop-loss reinsurance contract for catastrophic events. We achieve an absence of arbitrage opportunities in the market by using an equivalent martingale probability measure in the pricing model for catastrophe reinsurance contract. The Esscher transform is employed to change the probability measure.

KEYWORDS

Doubly stochastic Poisson process. Shot noise process. Piecewise deterministic Markov process theory. Stop-loss reinsurance contract. Equivalent martingale probability measure. Esscher transform.

1. INTRODUCTION

Insurance companies have traditionally used reinsurance contracts to hedge themselves against losses from catastrophic events. During the last decade, the world has experienced a higher level of catastrophic events both in terms of frequency and severity. Some of the recent catastrophes are Hurricane Andrew (USA 1992) and the Kobe earthquake (Japan 1995) (see Booth (1997)). This has had a marked effect on the reinsurance market. Such events have impacted the profitability and capital bases of reinsurance companies, some of which have withdrawn from the market, and others have reduced the level of catastrophe cover they are willing to provide.

In the early 1990s, some believed that there was undercapacity provided by the reinsurance market. Some investment banks, particularly US banks, recognised the opportunities that existed in the reinsurance market. Through their large capital bases the investment banks were able to offer alternative reinsurance products. This caused reinsurance companies to assess their strategies for the type of products offered to the market and emphasised the need for an appropriate pricing model for reinsurance contracts.

Let $\aleph_i$ be the claim amount, which are assumed to be independent and identically distributed with distribution function $H(u)$ $(u > 0)$ then the total loss excess over b, which is a retention limit, up to time t is

$$\left(\sum_{i=1}^{N_t} \aleph_i - b\right)^+ \tag{1.1}$$

where N_t is the number of claims up to time t and $\left(\sum_{i=1}^{N_t} \aleph_i - b\right)^+ = Max\left(\sum_{i=1}^{N_t} \aleph_i - b, 0\right)$.

Let $C_t = \sum_{i=1}^{N_t} \aleph_i$ be the total amount of claims up to time t. Then

$$\left(\sum_{i=1}^{N_t} \aleph_i - b\right)^+ = (C_t - b)^+. \tag{1.2}$$

Therefore the stop-loss reinsurance premium at time 0 is

$$E\{(C_t - b)^+\} \tag{1.3}$$

where the expectation is calculated under an appropriate probability measure. Throughout the paper, for simplicity, we assume interest rates to be constant.

In insurance modelling, the Poisson process has been used as a claim arrival process. Extensive discussion of the Poisson process, from both applied and theoretical viewpoints, can be found in Cramér (1930), Cox & Lewis (1966), Bühlmann (1970), Cinlar (1975), and Medhi (1982). However, there has been a significant volume of literature that questions the appropriateness of the Poisson process in insurance modelling (see Seal (1983) and Beard *et al.*(1984)) and more specifically for rainfall modelling (see Smith (1980) and Cox & Isham (1986)).

For catastrophic events, the assumption that resulting claims occur in terms of the Poisson process is inadequate. Therefore an alternative point process needs to be used to generate the claim arrival process. We will employ a doubly stochastic Poisson process, or the Cox process, (see Cox (1955), Bartlett (1963), Serfozo (1972), Grandell (1976, 1991), Bremaud (1981) and Lando (1994)).

The shot noise process can be used as the parameter of doubly Stochastic Poisson process to measure the number of claims due to catastrophic event (see Cox & Isham (1980,1986) and Klüppelberg & Mikosch (1995)). As time passes, the shot noise process decreases as more and more claims are settled. This decrease continues until another catastrophe occurs which will result in a positive jump in the shot noise process. The shot noise process is particularly useful in the claim arrival process as it measures the frequency, magnitude and time period needed to determine the effect of the catastrophic events. Therefore we will use it as a claim intensity function to generate doubly stochastic Poisson process. We will adopt the shot noise process used by Cox & Isham (1980):

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\substack{\text{all } i \\ s_i < t}} y_i e^{-\delta(t-s_i)}$$

where:

i catastrophe

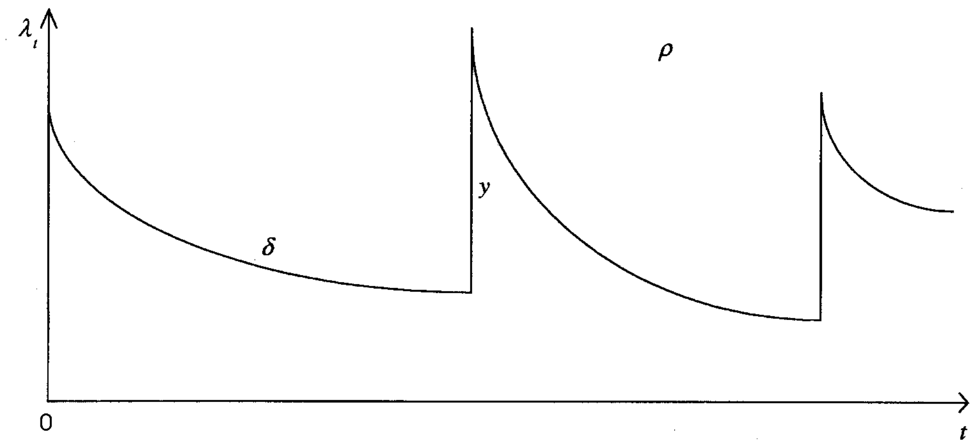
λ_0 initial value of λ

y_i jump size of catastrophe i (i.e. magnitude of contribution of catastrophe i to intensity) where $E(y_i) < \infty$

s_i time at which catastrophe i occurs where $s_i < t < \infty$

δ exponential decay which never reaches zero

ρ the number of catastrophes in time period t.



The piecewise deterministic Markov processes theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. We present definitions and important properties of the Cox and shot noise processes with the aid of piecewise deterministic processes theory (see Dassios (1987) and Dassios & Embrechts (1989)). This theory is used to calculate the distribution of the number of claims and the mean of the number of claims. These are important factors in the pricing of any reinsurance product.
We use the asymptotic (stationary) distribution of the claim intensity to derive pricing model for a stop-loss reinsurance contract for catastrophic events. Their application in computing the premiums will be illustrated.
Harrison & Kreps (1979) and Harrison & Pliska (1981) launched the approach for the pricing and analysis of movements of the financial derivatives whose prices are determined by the price of the underlying assets. Their mathematical framework originates from the idea of risk-neutral, or non-arbitrage, valuation of Cox & Ross (1976).
A reinsurance contract is similar to a financial derivative in that its value is determined by the underlying claim arrival process. Sondermann (1991) introduced the non-arbitrage approach for the pricing of reinsurance contracts. He proved that if there is no arbitrage opportunities in the market, reinsurance premiums are calculated by the expectation of their value at maturity with respect to a new probability measure and not with respect to the original probability measure. This new probability measure is called the equivalent martingale probability measure. Cummins & Geman (1995) also employed the non-arbitrage pricing technique. Besides, Aase (1994) and Embrechts & Meister (1995) discussed pricing techniques such as the general equilibrium approach and the utility maximisation pricing.
If the underlying stochastic process is not involved with jump structure (so called complete case), the fair price of a contingent claim is the expectation with respect to exactly one equivalent martingale probability measure (i.e. by assuming that there is an absence of arbitrage opportunities in the market). However doubly stochastic Poisson process has jump structures, we lose completeness and there will be infinitely many equivalent martingale probability measures. It depends on insurance companies' attitude towards to risk which equivalent martingale probability measure should be used. Therefore it is not the purpose of this paper to decide which one to use.
One of the methods to change the probability measure is the Esscher transform. Gerber & Shiu (1996) priced derivatives using the Esscher transform to go from the original probability

measure to the equivalent martingale probability measure. We use this approach for the pricing of stop-loss reinsurance contracts for catastrophic events.

2. DOUBLY STOCHASIC POISSON PROCESS AND SHOT NOISE PROCESS

Under doubly stochastic Poisson process, or the Cox process, the claim intensity function is assumed to be stochastic. The Cox process is more appropriately used as a claim arrival process as catastrophic events should be based on a specific stochastic process. However, little work has been done to further develop this assumption in an insurance context. We will now proceed to examine the doubly stochastic Poisson process as the claim arrival process.
The doubly stochastic Poisson process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. Therefore the doubly stochastic Poisson process can be viewed as a two step randomisation procedure. A process λ_t is used to generate another process N_t by acting as its intensity. That is, N_t is a Poisson process conditional on λ_t which itself is a stochastic process (if λ_t is deterministic then N_t is a Poisson process).
Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one adopted by Bremaud (1981).

Definition 2.1 *Let (Ω, F, P) be a probability space with information structure F. The information structure F is the filtration, i.e. $F = \{\Im_t, t \in [0,T]\}$. F consists of σ-algebra's $\Im_t$ on Ω, for any point t in the time interval [0, T], representing the information available at time t. Let N_t be a point process adopted to a history $\Im_t$ and let λ_t be a non- negative process. Suppose that λ_t is $\Im_t$-measurable, $t \geq 0$ and that*

$$\int_0^t \lambda_s ds < \infty \quad \textit{almost surely (no explosions).}$$

If for all $0 \leq t_1 \leq t_2$ and $u \in \Re$

$$E\left\{e^{iu(N_{t_2}-N_{t_1})} \middle| \Im_{t_2}\right\} = \exp\left\{\left(e^{iu}-1\right)\int_{t_1}^{t_2} \lambda_s ds\right\} \tag{2.1}$$

then N_t is called a $\Im_t$-doubly stochastic Poisson process with intensity λ_t.

□

(2.1) gives us

$$\Pr\left\{N_{t_2} - N_{t_1} = k \middle| \lambda_s; t_1 \leq s \leq t_2\right\} = \frac{e^{-\int_{t_1}^{t_2}\lambda_s ds}\left(\int_{t_1}^{t_2}\lambda_s ds\right)^k}{k!} \tag{2.2}$$

Now consider the process $X_t = \int_0^t \lambda_s ds$ *(the aggregated process)*, then from (2.2) we can easily find that

$$E\left(\theta^{N_{t_2}-N_{t_1}}\right) = E\{e^{-(1-\theta)(X_{t_2}-X_{t_1})}\}. \tag{2.3}$$

(2.3) suggests that the problem of finding the distribution of N_t, the point process, is equivalent to the problem of finding the distribution of X_t, the aggregated process. It means that we just have to find the p.g.f. (probability generating function) of N_t to retrieve the m.g.f. (moment generating function) of X_t and vice versa.

The three parameters of the shot noise process described in the previous section are homogeneous in time. We are now going to generalise the shot noise process by allowing three parameters to depend on time. Therefore we assume that $\delta(t)$, $\rho(t)$ and $G(y;t)$ are all Riemann integrable functions of t and are all positive. Furthermore the rate of jump arrivals, $\rho(t)$, is bounded on all intervals [0, t) (no explosions). $\delta(t)$ is the rate of decay and the distribution function of jump sizes for all t is $G(y;t)$ $(y>0)$. If the jump size distribution is exponential, its density is $g(y;t)=(\alpha+\gamma e^{\delta t})\exp\{-(\alpha+\gamma e^{\delta t})y\}$, $y>0$, $\alpha+\gamma e^{\delta t}>0$ (i.e. $\gamma>-\alpha e^{-\delta t}$), a special case that will be quite useful later.

If λ_t is a Markov process the generator of the process (λ_t, t) acting on a function $f(\lambda,t)$ belonging to its domain is given by

$$\mathrm{A}\, f(\lambda,t)=\frac{\partial f}{\partial t}-\delta(t)\lambda\frac{\partial f}{\partial \lambda}+\rho(t)\{\int_0^{\infty} f(\lambda+y,t)dG(y;t)-f(\lambda,t)\}. \tag{2.4}$$

The generator of the process (X_t,λ_t,t) acting on a function $f(x,\lambda,t)$ belonging to its domain is given by

$$\mathrm{A}\, f(x,\lambda,t)=\frac{\partial f}{\partial t}+\lambda\frac{\partial f}{\partial x}-\delta(t)\lambda\frac{\partial f}{\partial \lambda}+\rho(t)[\int_0^{\infty} f(x,\lambda+y,t)dG(y;t)-f(x,\lambda,t)]. \tag{2.5}$$

Also the generator of (N_t,λ_t,t) acting on $f(n,\lambda,t)$ is given by

$$\mathrm{A}\, f(n,\lambda,t)=\frac{\partial f}{\partial t}+\lambda[f(n+1,\lambda,t)-f(n,\lambda,t)]-\delta(t)\lambda\frac{\partial f}{\partial \lambda}+\rho(t)[\int_0^{\infty} f(n,\lambda+y,t)dG(y;t)-f(n,\lambda,t)]. \tag{2.6}$$

For $f(x,n,\lambda,t)$ to belong to the domain of the generator A, it is sufficient that $f(x,n,\lambda,t)$ is differentiable w.r.t. x, λ, t for all x, n, λ, t and that $\left|\int_0^{\infty} f(\cdot,\lambda+y,\cdot)dG(y;t)-f(\cdot,\lambda,\cdot)\right|<\infty$.

Let us evaluate the Laplace transform of the distribution of X_t, N_t and λ_t at time t.

Theorem 2.2 *Let X_t and λ_t be as defined. Also consider constants k and v such that $k\geq 0$ and $v\geq 0$, then*

$$\exp(-vX_t)\exp[-\{ke^{\Delta(t)}-ve^{\Delta(t)}\int_0^t e^{-\Delta(r)}dr\}\lambda_t]\exp[\int_0^t \rho(s)[1-\hat{g}\{ke^{\Delta(s)}-ve^{\Delta(s)}\int_0^s e^{-\Delta(r)}dr;s\}]ds] \tag{2.7}$$

is a martingale where $\hat{g}(u;s)=\int_0^{\infty} e^{-uy}dG(y;s)$ *and* $\Delta(t)=\int_0^t \delta(s)ds$.

Proof

From (2.5) $f(x,\lambda,t)$ has to satisfy $\mathrm{A}f=0$ for it to be a martingale. Setting $f=e^{-vx}e^{-A(t)\lambda}e^{R(t)}$ we get the equation

$$-\lambda A'(t)+R'(t)-\lambda v+\delta(t)\lambda A(t)+\rho(t)[\hat{g}\{A(t);t\}-1]=0 \tag{2.8}$$

and solving (2.8) we get

$$A(t)=ke^{\Delta(t)}-\nu e^{\Delta(t)}\int_0^t e^{-\Delta(r)}dr \quad\text{and}\quad R(t)=\int_0^t \rho(s)[1-\hat{g}\{ke^{\Delta(s)}-\nu e^{\Delta(s)}\int_0^s e^{-\Delta(r)}dr;s\}]ds.$$

Put $\Delta(t)=\int_0^t \delta(s)ds$ and the result follows.

□

Let us assume that $\delta(t)=\delta$ throughout the rest of this paper.

Corollary 2.3 *Let X_t, N_t and λ_t be as defined. Also let $\nu_1\geq 0$, $\nu_2\geq 0$, $\nu\geq 0$, $0\leq\theta\leq 1$. Then*

$$E\{e^{-\nu_1(X_{t_2}-X_{t_1})}e^{-\nu_2\lambda_{t_2}}\,|\,X_{t_1},\lambda_{t_1}\}$$

$$=\exp[-\{\tfrac{\nu_1}{\delta}+(\nu_2-\tfrac{\nu_1}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}]\exp[-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\tfrac{\nu_1}{\delta}+(\nu_2-\tfrac{\nu_1}{\delta})e^{-\delta(t_2-s)};s\}]ds] \tag{2.9}$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}e^{-\nu\lambda_{t_2}}\,|\,N_{t_1},\lambda_{t_1}\}$$

$$=\exp[-\{\tfrac{1-\theta}{\delta}+(\nu-\tfrac{1-\theta}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}]\exp[-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\tfrac{1-\theta}{\delta}+(\nu-\tfrac{1-\theta}{\delta})e^{-\delta(t_2-s)};s\}]ds] \tag{2.10}$$

Proof

(2.9) follows immediately where we set $\nu=\nu_1$, $k=\dfrac{\nu_1}{\delta}+(\nu_2-\dfrac{\nu_1}{\delta})e^{-\delta t_2}$ in theorem 2.2. (2.10) follows from (2.9) and (2.3).

□

Corollary 2.4 *Let X_t, N_t and λ_t be as defined. Then*

$$E\{e^{-\nu_2\lambda_{t_2}}\,|\,\lambda_{t_1}\}=\exp\{-\nu\lambda_{t_1}e^{-\delta(t_2-t_1)}\}\exp[-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\nu e^{-\delta(t_2-s)};s\}]ds], \tag{2.11}$$

$$E\{e^{-\nu(X_{t_2}-X_{t_1})}\,|\,\lambda_{t_1}\}=\exp[-\tfrac{\nu}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}]\exp[-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\tfrac{\nu}{\delta}(1-e^{-\delta(t_2-s)});s\}]ds] \tag{2.12}$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}\,|\,\lambda_{t_1}\}=\exp[-\tfrac{1-\theta}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}]\exp[-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\tfrac{1-\theta}{\delta}(1-e^{-\delta(t_2-s)});s\}]ds]. \tag{2.13}$$

Proof

If we set $\nu_1=0$ in (2.9) then (2.11) follows. If we also set $\nu_2=0$, $\nu=0$ in (2.9) and (2.10) then (2.12) and (2.13) follow.

□

We can obtain the asymptotic (stationary) distributions of λ_t at time t from (2.11), provided that the process started sufficiently far in the past. In this context we interpret it as the limit when $t\to-\infty$. In other words, if we know λ at '$-\infty$' and no information between '$-\infty$' to present time t, '$-\infty$' asymptotic distribution of λ_t can be used as the distribution of λ_t. It is easy to check that if $\delta(t)=\delta$, $\lim_{t\to-\infty}\rho(t)=\rho$ and $\lim_{t\to-\infty}\mu_1(t)=\mu_1$ then

$$\int_{-\infty}^{t}\rho(s)[1-\hat{g}\{ve^{-\delta(t-s)};s\}]ds<\infty \tag{2.14}$$

where $\mu_1(t)=\int_0^{\infty} ydG(y;t)=E(y;t)$ and $\hat{G}(u;t)=\dfrac{1-\hat{g}(u;t)}{u}$.

Lemma 2.5 *Let λ_t be as defined. Also let $\delta(t)=\delta$ and assume that $\lim_{t\to-\infty}\rho(t)=\rho$ and $\lim_{t\to-\infty}\mu_1(t)=\mu_1$. Then the '$-\infty$' asymptotic distribution of λ_t has Laplace transform*

$$E(e^{-v\lambda_{t_1}})=\exp[-\int_{-\infty}^{t_1}\rho(s)[1-\hat{g}\{ve^{-\delta(t_1-s)};s\}]ds] \tag{2.15}$$

Proof

Without loss of generality, if we change the time scale in (2.11),

$$E\left\{e^{-v_2\lambda_{t_1}}\middle|\lambda_{t_0}\right\}=\exp\{-v\lambda_{t_0}e^{-\delta(t_1-t_0)}\}\exp[-\int_{t_0}^{t_1}\rho(s)[1-\hat{g}\{ve^{-\delta(t_1-s)};s\}]ds]. \tag{2.16}$$

Let $t_0\to-\infty$ in (2.16) then the result follows immediately.

□

Theorem 2.6 *Let X_t, N_t and λ_t be as defined and the jump size distribution be exponential i.e. $g(y;t)=(\alpha+\gamma e^{\delta t})\exp\{-(\alpha+\gamma e^{\delta t})y\}$, $y>0$, $\gamma>-\alpha e^{-\delta t}$. Assuming that $\rho(t)=\rho\dfrac{\alpha}{\alpha+\gamma e^{\delta t}}$⁺ then*

$$E\left\{e^{-v\lambda_{t_1}}\middle|\lambda_{t_0}\right\}=\exp\{-v\lambda_0e^{-\delta(t_1-t_0)}\}\left(\frac{\gamma e^{\delta t_0}+\alpha e^{-\delta(t_1-t_0)}}{\gamma e^{\delta t_0}+\alpha}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma e^{\delta t_0}+ve^{-\delta(t_1-t_0)}+\alpha}{\gamma e^{\delta t_0}+(v+\alpha)e^{-\delta(t_1-t_0)}}\right)^{\frac{\rho}{\delta}}, \tag{2.17}$$

$$E\{e^{-v(X_{t_2}-X_{t_1})}\big|\lambda_{t_1}\}=\exp[-\tfrac{v}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}]\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}} \tag{2.18}$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}\big|\lambda_{t_1}\}=\exp[-\tfrac{1-\theta}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}]\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}. \tag{2.19}$$

If λ_t is '$-\infty$' asymptotic,

$$E(e^{-v\lambda_{t_1}})=\left(\frac{\gamma+\alpha e^{-\delta t_1}}{\gamma+(v+\alpha)e^{-\delta t_1}}\right)^{\frac{\rho}{\delta}}, \tag{2.20}$$

⁺ Note: The reason for this particular assumption will become apparent later when we change the probability measure.

$$E\{e^{-\nu(X_{t_2}-X_{t_1})}\}=\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha+\frac{\nu}{\delta}(1-e^{-\delta(t_2-t_1)})}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{\nu}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+\nu}} \quad (2.21)$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}\}=\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}$$

(2.22)

Proof

If we set $\rho(t)=\rho\frac{\alpha}{\alpha+\gamma e^{\delta t}}$ and $g(y;t)=(\alpha+\gamma e^{\delta t})\exp\{-(\alpha+\gamma e^{\delta t})y\}$, $y>0$, $\gamma>-\alpha e^{-\delta t}$ in (2.11), (2.12) and (2.13) then (2.17), (2.18) and (2.19) follow.

Let $t_0\to-\infty$ in (2.17) then (2.20) follows immediately, from which (2.21) and (2.22) follow.

□

Now let us derive the expected value of N_t.

Theorem 2.7 *Let N_t and λ_t be as defined. Consider constants γ^*, θ^* such that $\gamma^*\le 0$ and $\theta^*\ge 1$. Then*

$$E(N_{t_2}-N_{t_1})=\int_{t_1}^{t_2}E(\lambda_s)ds=\left(\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right)E(\lambda_{t_1})+\frac{1}{\delta}\int_{t_1}^{t_2}(1-e^{-\delta(t_2-s)})\rho(s)\mu_1(s)ds. \quad (2.23)$$

If the jump size distribution be exponential, i.e. $g(y;t)=(\alpha+\gamma e^{\delta t})\exp\{-(\alpha+\gamma e^{\delta t})y\}$, $y>0$, $\gamma>-\alpha e^{-\delta t}$ *with* $\rho(t)=\rho\frac{\alpha}{\alpha+\gamma e^{\delta t}}$ *and λ_t is '$-\infty$' asymptotic*

$$E(N_{t_2}-N_{t_1})=\frac{\rho}{\delta\alpha}(t_2-t_1)-\frac{\rho}{\delta^2\alpha}\ln\left(\frac{\gamma e^{\delta t_2}+\alpha}{\gamma e^{\delta t_1}+\alpha}\right). \quad (2.24)$$

Proof

Using (2.4), we can obtain

$$E\left(\lambda_{t_1}\middle|\lambda_{t_0}\right)=\lambda_{t_0}e^{-\delta(t_1-t_0)}+e^{-\delta t_1}\int_{t_0}^{t_1}e^{\delta s}\rho(s)\mu_1(s)ds \quad (2.25)$$

and by letting $t_0\to-\infty$ in (2.25), we can obtain the '$-\infty$' asymptotic expected value of λ_t;

$$E(\lambda_{t_1})=e^{-\delta t_1}\int_{-\infty}^{t_1}e^{\delta s}\rho(s)\mu_1(s)ds \quad (2.26)$$

where $\mu_1(t)=\int_0^\infty y\,dG(y;t)=E(y;t)$. From (2.2)

$$E[N_{t_2} - N_{t_1}] = \int_{t_1}^{t_2} E(\lambda_s)ds. \quad (2.27)$$

Condition on λ_{t_1} in (2.27) and use (2.25) then (2.23) follows immediately.

If the jump size distribution is exponential i.e. $g(y;t) = (\alpha + \gamma e^{\delta t})\exp\{-(\alpha + \gamma e^{\delta t})y\}$, $y > 0$, $\gamma > -\alpha e^{-\delta t}$ and $\rho(t) = \rho \dfrac{\alpha}{\alpha + \gamma e^{\delta t}}$ the '$-\infty$' asymptotic expected value of λ_t becomes

$$E(\lambda_{t_1}) = \frac{\rho}{\delta(\alpha + \gamma e^{\delta t_1})}. \quad (2.28)$$

Therefore set $\rho(s) = \rho \dfrac{\alpha}{\alpha + \gamma e^{\delta s}}$, $\mu_1(s) = \dfrac{1}{\alpha + \gamma e^{\delta s}}$ in (2.23) and use (2.28), then (2.24) follows immediately.

□

3. THE ESSCHER TRANSFORM AND CHANGE OF PROBABILITY MEASURE

The assumption of no arbitrage opportunities in the market is equivalent to the existence of an equivalent martingale probability measure. We will examine an equivalent martingale probability measure obtained via the Esscher transform (see Gerber & Shiu, 1996). In general, the Esscher transform is defined as a change of probability measure for certain stochastic processes. An Esscher transform of such a process induces an equivalent probability measure on the process. The parameters involved for an Esscher transform are determined so that the price of a random payment in the future is a martingale under the new probability measure. A random payment therefore is calculated as the expectation of that at maturity with respect to the equivalent martingale probability measure (also known as the risk-neutral Esscher measure).

We here offer the definition of the Esscher transform that is adopted from Gerber & Shiu (1996).

Definition 3.1 *Let X_t be a stochastic process and h^* a real number. For a measurable function f, the expectation of the random variable $f(X_t)$ with respect to the equivalent martingale probability measure is*

$$E^*[f(X_t)] = E\left[f(X_t)\frac{e^{h^*X_t}}{E\left(e^{h^*X_t}\right)}\right] = \frac{E\left[f(X_t)e^{h^*X_t}\right]}{E\left[e^{h^*X_t}\right]} \quad (3.1)$$

where the process $\dfrac{e^{h^*X_t}}{E\left(e^{h^*X_t}\right)}$ is a martingale.

□

From definition 3.1, we need to obtain a martingale that can be used to define a change of probability measure, i.e. it can be used to define the Radon-Nikodym derivative $\dfrac{dP^*}{dP}$ where P is the original probability measure and P^* is the equivalent martingale probability measure with parameters involved. This martingale will be used to calculate the premiums for stop-loss reinsurance contract. Furthermore, using this equivalent martingale probability measure, the pricing models for stop-loss reinsurance contract will be established and illustrated through numerical examples. In general, more than one equivalent martingale probability measure exists but it is not the purpose of this paper to decide which is the appropriate one to use.

Let M_t be the total number of catastrophe jumps up to time t. We will assume that claim points and catastrophe jumps do not occur at the same time.

The generator of the process $(X_t, N_t, C_t, \lambda_t, M_t, t)$ acting on a function $f(x,n,c,\lambda,m,t)$ belonging to its domain is given by

$$\begin{aligned} \mathrm{A}\, f(x,n,c,\lambda,m,t) &= \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial x} + \lambda\Big[\int_0^\infty f(x,n+1,c+u,\lambda,m,t)dH(u) - f(x,n,c,\lambda,m,t)\Big] \\ &\quad - \delta\lambda \frac{\partial f}{\partial \lambda} + \rho\Big[\int_0^\infty f(x,n,c,\lambda+y,m+1,t)dG(y) - f(x,n,c,\lambda,m,t)\Big]. \end{aligned} \tag{3.2}$$

Clearly, for $f(x,n,c,\lambda,m,t)$ to belong to the domain of the generator A, it is essential that $f(x,n,c,\lambda,m,t)$ is differentiable w.r.t. x, c, λ, t for all x, n, c, λ, m, t and that $\left|\int_0^\infty f(\cdot,\lambda+y,\cdot)dG(y) - f(\cdot,\lambda,\cdot)\right| < \infty$ and $\left|\int_0^\infty f(\cdot,c+u,\cdot)dH(u) - f(\cdot,c,\cdot)\right| < \infty$.

Lemma 3.2 *Let λ_t as defined. Assume that $f(n,\lambda,t) = f(\lambda,t)$ for all n and that $e^{v^*\lambda_t}$ is a martingale. Consider a constant v^* such that $v^* \geq 0$. Then*

$$\mathrm{A}^* f(\lambda,0) = \frac{\mathrm{A}\,\{f(\lambda,0)e^{-v^*\lambda}\}}{e^{-v^*\lambda}}. \tag{3.3}$$

<u>**Proof**</u>

The generator of the process (λ_t, t) acting on a function $f(\lambda,t)$ with respect to the equivalent martingale probability measure is

$$\mathrm{A}^* f(\lambda,0) = \lim_{t\downarrow 0} \frac{E^*[f(\lambda_t,t)|\lambda_0 = \lambda] - f(\lambda,0)}{t}. \tag{3.4}$$

We will use $\dfrac{e^{v^*\lambda_t}}{E(e^{v^*\lambda_t})}$ as the Radon-Nikodym derivative to define equivalent martingale probability measure. Hence, the expected value of $f(\lambda_t,t)$ given λ with respect to the equivalent martingale probability measure is

$$E^*\{f(\lambda_t,t)|\lambda_0 = \lambda\} = \frac{E[f(\lambda_t,t)\cdot e^{-v^*\lambda_t}|\lambda_0 = \lambda]}{E(e^{-v^*\lambda_t}|\lambda_0 = \lambda)}. \tag{3.5}$$

Since the denominator in (3.5) is a martingale, it becomes

$$E^*\{f(\lambda_t,t)|\lambda_0 = \lambda\} = \frac{f(\lambda,0)\cdot e^{-v^*\lambda} + \int_0^t E[\mathrm{A}\, f(\lambda_s,s)\cdot e^{-v^*\lambda_s}|\lambda_0 = \lambda]ds}{e^{-v^*\lambda}}. \tag{3.6}$$

Set (3.6) in (3.4) then

$$\mathrm{A}^* f(\lambda,0) = \frac{1}{e^{-v^*\lambda}} \lim_{t\downarrow 0} \int_0^t \frac{E[\mathrm{A}\, f(\lambda_s,s)\cdot e^{-v^*\lambda_s}|\lambda_0 = \lambda]ds}{t}. \tag{3.7}$$

Therefore, from Dynkin's formula (see Øksendal (1992)) (3.3) follows immediately.

□

Theorem 3.3 *Let N_t, C_t, λ_t and M_t be as defined. Consider constants θ^*, v^*, ψ^* and γ^*, such that $\theta^* \geq 1$, $v^* \leq 0$, $\psi^* \geq 1$ and $\gamma^* \leq 0$. Then*

$$\theta^{*N_t} e^{-v^* C_t} \exp[-\{\theta^* \hat{h}(v^*) - 1\}\int_0^t \lambda_s ds] \psi^{*M_t} \exp(-\gamma^* \lambda_t e^{\delta t}) \exp[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds] \quad (3.8)$$

is a martingale.

Proof

From (3.2), $f(x,n,c,\lambda,m,t)$ has to satisfy $\mathrm{A}f = 0$ for $f(X_t, N_t, C_t, \lambda_t, M_t, t)$ to be a martingale. Trying $\theta^{*n} e^{-v^* c} e^{\phi^* x} \psi^{*m} \exp(-\gamma^* \lambda e^{\delta t}) e^{A(t)}$ we get the equation

$$A'(t) + \lambda\phi^* + \lambda\{\theta^* \hat{h}(v^*) - 1\} + \rho\{\psi^* \hat{g}(\gamma^* e^{\delta t}) - 1\} = 0 \quad (3.9)$$

and solving (3.9) we get

$$\phi^* = -\{\theta^* \hat{h}(v^*) - 1\} \quad \text{and} \quad A(t) = \rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds$$

and the result follows.

□

Now, let us examine the generator A^* of the process $(X_t, N_t, C_t, \lambda_t, M_t, t)$ acting on a function $f(x,n,c,\lambda,m,t)$ with respect to the equivalent martingale probability measure .

Theorem 3.4 *Let N_t, C_t, λ_t and M_t be as defined. Consider constants θ^*, v^*, ψ^* and γ^*, such that $\theta^* \geq 1$, $v^* \leq 0$, $\psi^* \geq 1$ and $\gamma^* \leq 0$. Then*

$$\mathrm{A}^* f(x,n,c,\lambda,m,t) = \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial x} + \theta^* \hat{h}(v^*) \lambda \{\int_0^\infty f(x, n+1, c+u, \lambda, m, t) dH^*(u) - f(x,n,c,\lambda,m,t)\}$$
$$- \delta\lambda \frac{\partial f}{\partial \lambda} + \rho^*(t) \{\int_0^\infty f(x,n,c,\lambda+y, m+1, t) dG^*(y;t) - f(x,n,c,\lambda,m,t)\} \quad (3.10)$$

where $dH^*(u) = \dfrac{e^{-v^* u} dH(u)}{\hat{h}(v^*)}$, $\rho^*(t) = \rho\psi^* \hat{g}(\gamma^* e^{\delta t})$ *and* $dG^*(y;t) = \dfrac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$.

Proof

From theorem 3.3, we can use

$$\frac{\theta^{*N_t} e^{-v^* C_t} \exp[-\{\theta^* \hat{h}(v^*) - 1\}\int_0^t \lambda_s ds] \psi^{*M_t} \exp(-\gamma^* \lambda_t e^{\delta t}) \exp[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds]}{E[\theta^{*N_t} e^{-v^* C_t} \exp[-\{\theta^* \hat{h}(v^*) - 1\}\int_0^t \lambda_s ds] \psi^{*M_t} \exp(-\gamma^* \lambda_t e^{\delta t}) \exp[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds]]} \quad (3.11)$$

as the Radon-Nikodym derivative to define an equivalent martingale probability measure.

Therefore from lemma 3.2

$\mathrm{A}^* f(X_t, N_t, C_t, \lambda_t, M_t, t)$

$$= \frac{\mathrm{A}\, f(X_t, N_t, C_t, \lambda_t, M_t, t) \cdot \theta^{*N_t} e^{-v^* C_t} \exp[-\{\theta^* \hat{h}(v^*) - 1\}\int_0^t \lambda_s ds] \psi^{*M_t} \exp(-\gamma^* \lambda_t e^{\delta t}) \exp[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds]}{\theta^{*N_t} e^{-v^* C_t} \exp[-\{\theta^* \hat{h}(v^*) - 1\}\int_0^t \lambda_s ds] \psi^{*M_t} \exp(-\gamma^* \lambda_t e^{\delta t}) \exp[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds]}.$$

From (3.2), using the generator with respect to the original probability measure,

$$\mathcal{A}\, f(x,n,c,\lambda,m,t)\theta^{*N_t} e^{-v^* C_t} \exp[-\{\theta^* \hat{h}(v^*) - 1\}\int_0^t \lambda_s ds]\psi^{*M_t} \exp(-\gamma^* \lambda_t e^{\delta t})\exp[\rho\int_0^t \{1-\psi^* \hat{g}(\gamma^* e^{\delta s})\}ds]$$

$$= [\frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} + \lambda\{\theta^* \int_0^\infty f(x,n+1,c+u,\lambda,m,t)e^{-v^* u}dH(u) - \theta^* \hat{h}(v^*)f(x,n,c,\lambda,m,t)\}$$

$$-\delta\lambda\frac{\partial f}{\partial \lambda} + \rho\{\psi^* \int_0^\infty f(x,n,c,\lambda+y,m+1,t)dG(y) - \psi^* \hat{g}(\gamma^* e^{\delta t})f(x,n,c,\lambda,m,t)\}]$$

$$\cdot\theta^{*N_t} e^{-v^* C_t} \exp[-\{\theta^* \hat{h}(v^*) - 1\}\int_0^t \lambda_s ds]\psi^{*M_t} \exp(-\gamma^* \lambda_t e^{\delta t})\exp[\rho\int_0^t \{1-\psi^* \hat{g}(\gamma^* e^{\delta s})\}ds].$$

Therefore

$$\mathcal{A}^* f(x,n,c,\lambda,m,t) = \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} + \theta^* \hat{h}(v^*)\lambda\{\int_0^\infty f(x,n+1,c+u,\lambda,m,t)dH^*(u) - f(x,n,c,\lambda,m,t)\}$$

$$-\delta\lambda\frac{\partial f}{\partial \lambda} + \rho^*(t)\{\int_0^\infty f(x,n,c,\lambda+y,m+1,t)dG^*(y;t) - f(x,n,c,\lambda,m,t)\} \tag{3.12}$$

where $dH^*(u) = \dfrac{e^{-v^* u}dH(u)}{\hat{h}(v^*)}$, $\rho^*(t) = \rho\psi^* \hat{g}(\gamma^* e^{\delta t})$ *and* $dG^*(y;t) = \dfrac{\exp(-\gamma^* e^{\delta t} y)dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$.

□

Theorem 3.4 yields the following:

(i) The claim intensity function λ_t has changed to $\lambda_t \theta^* \hat{h}(v^*)$;

(ii) The rate of jump arrival ρ has changed to $\rho^*(t) = \rho\psi^* \hat{g}(\gamma^* e^{\delta t})$ (it now depends on time);

(iii) The jump size measure $dG(y)$ has changed to $dG^*(y;t) = \dfrac{\exp(-\gamma^* e^{\delta t} y)dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$ (it now depends on time);

(iv) The claim size measure $dH(u)$ has changed to $dH^*(u) = \dfrac{e^{-v^* x}dH(u)}{\hat{h}(v^*)}$.

Let us evaluate the '$-\infty$' asymptotic expected value of N_t and the Laplace transform of the '$-\infty$' asymptotic distribution of N_t with respect to the equivalent martingale probability measure, i.e. $E^*(N_t)$ and $E^*(\theta^{N_t})$. We will assume that the jump size distribution is exponential ($g(y) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$) and that λ_t is '$-\infty$' asymptotic. Therefore we can obtain that $g^*(y;t) = (\alpha + \gamma^* e^{\delta t})\exp\{-(\alpha + \gamma^* e^{\delta t})y\}$, $y > 0$, $-\alpha e^{-\delta t} < \gamma^* \le 0$ and $t < \frac{1}{\delta}\ln(-\frac{\alpha}{\gamma^*})$ since $dG^*(y;t) = \dfrac{\exp(-\gamma^* e^{\delta t} y)dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$. It is clear that such a model is appropriate in the short term only, as it breaks down for $t \ge \frac{1}{\delta}\ln(-\frac{\alpha}{\gamma^*})$. For simplicity, let us assume that $v^* = 0$ and $\psi^* = 1$.

Corollary 3.5 *Let N_t be as defined and the jump size distribution be exponential. Consider constants θ, θ^*, v^*, ψ^* and γ^* such that $0 \le \theta \le 1$, $\theta^* \ge 1$, $v^* = 0$, $\psi^* = 1$ and $\gamma^* \le 0$. Furthermore if λ_t is '$-\infty$' asymptotic then*

$$E^*(\theta^{N_{t_2}-N_{t_1}}) = \left(\frac{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}}{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^*(1-\theta)}{\delta}(1-e^{-\delta(t_2-t_1)})} \right)^{\frac{\rho}{\delta}} \left(\frac{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^*(1-\theta)}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}} \right)^{\frac{\alpha\rho}{\delta\alpha + \theta^*(1-\theta)}} \tag{3.13}$$

and

$$E^*(N_{t_2} - N_{t_1}) = \frac{\theta^* \rho}{\delta \alpha}(t_2 - t_1) - \frac{\theta^* \rho}{\delta^2 \alpha} \ln\left(\frac{\gamma^* e^{\delta t_2} + \alpha}{\gamma^* e^{\delta t_1} + \alpha} \right) \tag{3.14}$$

where $0 < t_1 < t_2 < t$.

Proof

From theorem 3.4 and (2.3) $E^*(\theta^{N_{t_2}-N_{t_1}}) = E[\exp\{-\theta^* \hat{h}(v^*)(1-\theta)\int_{t_1}^{t_2} \lambda_s ds\}]$ where $dH^*(u) = \frac{e^{-v^* u} dH(u)}{\hat{h}(v^*)}$, $\rho^*(t) = \rho \psi^* \hat{g}(\gamma^* e^{\delta t})$ and $dG^*(y;t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$. Since $v^* = 0$, $\psi^* = 1$ and the jump size distribution is exponential, $\rho^*(s) = \rho \frac{\alpha}{\alpha + \gamma^* e^{\delta s}}$ and $\mu_1^*(s) = \frac{1}{\alpha + \gamma^* e^{\delta s}}$. Therefore if we set $v = \theta^*(1-\theta)$ in (2.21) and multiply θ^* to (2.24), putting $\gamma = \gamma^*$, the results follow immediately.

□

4. PRICING OF A STOP-LOSS REINSURANCE CONTRACT FOR CATASTROPHIC EVENTS

Let us look at the stop-loss reinsurance premium at time 0 assuming that there is an absence of arbitrage opportunities in the market. This can be achieved by using an equivalent martingale probability measure, P^*, within the pricing model used for calculating premium for reinsurance contract. Therefore, from (1.3), the stop-loss reinsurance premium at time 0 is

$$E^*[(C_t - b)^+] \tag{4.1}$$

where all symbols have previously been defined.

We assume that the claim size distribution is gamma, i.e. $h(u) = \frac{\beta^\varphi u^{\varphi-1} e^{-\beta u}}{(\varphi-1)!}$, $u > 0$, $\beta > 0$, $\varphi \ge 1$. Then

$$E^*[(C_t - b)^+] = \sum_{n=1}^{\infty} a_n^* \{ \frac{n\varphi}{\beta} \int_b^{\infty} \frac{\beta^{n\varphi+1} c^{n\varphi} e^{-\beta c}}{(n\varphi)!} dc - b \int_b^{\infty} \frac{\beta^{n\varphi} c^{n\varphi-1} e^{-\beta c}}{(n\varphi-1)!} dc \} \tag{4.2}$$

where $a_n^* = P^*(N_t = n)$.

Let us illustrate the calculation of stop-loss reinsurance premiums for catastrophic events using the pricing models derived previously. The change of stop-loss reinsurance premiums associated with changes in value of θ^* and γ^* is also examined.

Let us also assume that the jump size distribution is exponential i.e. $g^*(y;t) = (\alpha + \gamma^* e^{\delta t})\exp\{-(\alpha + \gamma^* e^{\delta t})y\}$, $y > 0$, $-\alpha e^{-\delta t} < \gamma^* \le 0$ and $t < \frac{1}{\delta}\ln(-\frac{\alpha}{\gamma^*})$ and that λ_t is '$-\infty$' asymptotic. Consider constants θ^* and θ such that $\theta^* \ge 1$ and $0 \le \theta \le 1$.

From (3.13), the p.g.f. of N_t is

$$E^*(\theta^{N_t}) = \sum_{n=0}^{\infty} \theta \cdot P^*(N_t = n) = \sum_{n=0}^{\infty} \theta^n a_n^*$$

$$= \left(\frac{\gamma^* + \alpha e^{-\delta t}}{\gamma^* + \alpha + \frac{\theta^*(1-\theta)}{\delta}(1 - e^{-\delta t})} \right)^{\frac{\rho}{\delta}} \left(\frac{\gamma^* + \alpha + \frac{\theta^*(1-\theta)}{\delta}(1 - e^{-\delta t})}{\gamma^* + \alpha e^{-\delta t}} \right)^{\frac{\alpha\rho}{\delta\alpha + \theta^*(1-\theta)}} . \quad (4.3)$$

The parameter values used to expand (4.3) with respect to θ are

$$\theta^* = 1.1 \ , \ \gamma^* = -0.1 \ , \ \alpha = 1 \ , \ \delta = 0.3 \ , \ \rho = 4 \ , \ t = 1.$$

Using these parameter values we can calculate the mean of the claim number in a unit period of time. From (3.14)

$$E^*(N_t) = \frac{\theta^* \rho}{\delta \alpha} t - \frac{\theta^* \rho}{\delta^2 \alpha} \ln\left(\frac{\gamma^* e^{\delta t} + \alpha}{\gamma^* + \alpha} \right) \approx 16.61.$$

By expanding (4.3) using the *MAPLE* algebraic manipulations package we can obtain $a_n^* = P^*(N_t = n)$ which is as follows:

$$E^*(\theta^{N_t}) = \sum_{n=0}^{\infty} \theta^i \cdot P^*(N_t = n) = \sum_{n=0}^{\infty} \theta^n a_n^* = \left\{ \frac{0.64082}{0.9 + 0.95033(1-\theta)} \right\}^{\frac{4.4(1-\theta)}{0.09 + 0.33(1-\theta)}}$$

$= 0.000014982 + 0.00011628\theta + 0.00048266\theta^2 + 0.0014225\theta^3 + 0.0033355\theta^4 + 0.006615\theta^5 + 0.011523\theta^5 + 0.018086\theta^7 + 0.026045\theta^8 + 0.034881\theta^9 + 0.0439\theta^{11} + 0.052349\theta^{11} + 0.059537\theta^{12} + 0.064932\theta^{13} + 0.068214\theta^{14} + 0.06929\theta^{15} + 0.068273\theta^{16} + 0.065434\theta^{17} + 0.061148\theta^{18} + 0.055831\theta^{19} + 0.049898\theta^{20} + 0.043723\theta^{21} + 0.037616\theta^{22} + 0.031815\theta^{23} + 0.026484\theta^{24} + 0.02172\theta^{25} + 0.017567\theta^{26} + 0.014023\theta^{27} + 0.011056\theta^{28} + 0.0086166\theta^{29} + 0.0066419\theta^{30} + 0.0050667\theta^{31} + 0.0038272\theta^{32} + 0.0028639\theta^{33} + 0.0021241\theta^{34} + 0.0015621\theta^{35} + 0.0011396\theta^{36} + 0.00082497\theta^{37} + 0.00059282\theta^{38} + 0.00042301\theta^{39} + 0.00029981\theta^{40} + 0.00021112\theta^{41} + 0.00014775\theta^{42} + 0.00010279\theta^{43} + 0.000071101\theta^{44} + 0.000048911\theta^{45} + 0.000033469\theta^{46} + 0.000022785\theta^{47} + 0.000015436\theta^{48} + 0.000010407\theta^{49} + 0.000006985\theta^{50} + 0.0000046672\theta^{51} + 0.0000031051\theta^{52} + 0.0000020573\theta^{53} + 0.0000013575\theta^{54} + O(\theta^{55})$.

(4.4)

Example 4.1

The parameter values used to calculate (4.2) are

$$n: 1 \sim 41, \ \varphi = 1, \ \beta = 1, \ b = 0, 5, 10, 16.61, 20, 25, 30$$

$$E^*(C_t) = E^*(N_t)E(\aleph) = 16.61.$$

By computing (4.2) using *S-Plus* the calculation of the stop-loss reinsurance premiums for catastrophic events at each retention level b are shown in Table 4.1.

Table 4.1

Retention level b	Reinsurance premiums
0	16.58403
5	11.61916
10	7.06779
16.61	2.833487
20	1.587005
25	0.595824
30	0.1951147

Example 4.2

We will now examine the effect on stop-loss reinsurance premiums caused by changes in the value of θ^* and γ^*. By expanding (4.3) using *MAPLE* at each value of θ^* and γ^* respectively and computing (4.2) by *S-Plus*, the calculation of the stop-loss reinsurance premiums for catastrophic events at the retention limit $b = 25$ are shown in Table 4.2 and Table 4.3.

Table 4.2

θ^*	$\gamma^* = -0.1$
1.0	0.3544252
1.1	0.595824
1.2	0.9299355
1.3	1.366049
1.4	1.90885
1.5	2.558786

Table 4.3

γ^*	$\theta^* = 1.1$
0.0	0.3029752
-0.1	0.595824
-0.2	1.207256
-0.3	2.512553
-0.4	5.364622
-0.5	11.65184

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