Calabi-Yau Manifolds

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Chapter 1

Introduction

The aim of this paper is to introduce the reader to the world of Calabi-Yau manifolds. These are a special kind of complex manifold, which have the properties to be compact, Kähler and to have a Ricci-Flat metric. Thank to this, they happen to be of great interest for physicists, which presume the space-time fabric to be in part shaped like a Calabi-Yau manifold.
Our approach to Calabi-Yau manifolds is resolutely Differential Geometric:we regard them as smooth real manifolds equipped with a geometric structure. The alternative would be to define them using Algebraic Geometry, but this lies out of the main line of this paper.

In the set of all 2m dimensional real manifolds (we call this set A) there is only a part of it which admit a complex structure, i.e. which can be made complex manifolds by considering 2 real coordinates as real and imaginary part of a new complex coordinate. We will call this subset B.
There is another subset C, which contains all the complex manifolds having the property of being Kähler. These are the "nicest" class of metrics on a complex manifold, since their Riemannian structure is compatible with the complex structure.
The last subset D will consist of all these manifolds which admit a Ricci-flat metric, these are called Calabi-Yau Manifolds.
All these sets ar proper in the sense that, for example, there are complex manifolds which don't admit a Kähler metric, etc (there are manifolds in A but not in B, in B but not in C and so on). There is something to say also about the number of manifolds contained in each subset. As one could think, the set A of real manifolds and the set B of complex manifolds contain an infinite number of elements. The same is true for C, Kähler manifolds,

however it is believed that the set of Calabi-Yau manifolds be a finite one. Although it may seem weird to guess this number, a reasonable one may be 10'000 (Come?). In the following we shall see how to create large number of this manifolds, using the so called Calabi conjecture: a theorem that will be the center of our paper.

Having seen how all these subjects are related to each other, the best path to follow throughout the paper is the one that starts with real manifolds, mainly to define the notation but also to give the reader the opportunity to better follow the approach taken in chapter 3, which considers an m-dimensional complex manifolds as a 2m-dimensional real one. Thereafter we will come to the definition of complex structure and complex manifolds, showing examples of real manifolds that could or not can be made complex. After pausing on Kähler manifolds,it will be the turn of Calabi-Yau. In this chapter we will expose the Calabi conjecture, which can be used to construct a large number of Calabi-Yau manifolds. We shall then give a sketch of the differential geometrical aspect of the proof, given by Yau almost 20 years after the first publication of this theorem, in 1954. (Soggetto Lucio)
The last chapter will concentrate on the application of this mathematical field to physics. It will be explained why string physicists think that the space-time fabric would have the form of a four dimensional space with a six-dimensional Calabi-Yau manifold at each point. Here strings, supposed to be the building blocks of matter, would vibrate and appear to us like elementary particles.
In every chapter we shall try to illustrate things with an example; what we will try, is to find examples that fit in the subsets posed at the beginning of the introduction, namely to find, for every subset A,B,C,D, examples of manifolds which fit in one set but not in the next one, this in order to highlight the properties of each singular set.

Chapter 2

Real Manifolds

In order to introduce complex manifolds, we will first need to know something about the real ones. Being easier to understand, real manifolds will help us defining and visualizing concepts, which shall then be expanded to complex manifolds. This approach will be of particular interest in next chapter, where we will study complex m-dimensional manifolds as real 2m-dimensional ones. In This chapter we will focus our attention on two topics: the *curvature* and the *holonomy group* of a connection. We will see how restrictions upon the holonomy group gives rise to restriction upon the curvature of a manifold. This will be useful in future sections, where, studying the properties of Calabi-Yau manifolds, we will meet an example of such a relationship.

2.1 Real Manifolds and Forms

We shall start by defining real manifolds and by giving some backgrounds of differential geometry. This will be done mainly out of completeness and to fix notation, but also to prepare the reader for the next chapter, where we shall encounter complex definitions of manifold and tangent space.
We assume the reader is already familiar with the concept of submanifolds in $\mathbb{R}^n$, as zeroes of differentiable functions; such a definition has to be expanded to the case where our object is no longer embedded in some external space. For example, it is easy for us to visualize the earth surface as a sphere in $\mathbb{R}^3$; this wouldn't be the same if we were flat, lying on the surface of the sphere, without perception of the third dimension. In that case we would need a description with local coordinates, in the same way as we use maps to describe locally the earth surface.

A **smooth m-dimensional Real Manifold** M *is a Hausdorff space (where each two points have disjunct neighborhoods) with an abzaehlbare(?) basis of the topology and the property that for each point* $p \in M$ *there exists an injective map* $\varphi : U \to \varphi(U)$ *bringing an open neighborhood* U *of* p *to an open set* $\varphi(U) \subset \mathbb{R}^n$.

A point p on the manifold is then specified by **coordinates** $x^1(p), ..., x^n(p)$, the components of the function φ.
This combination of Neighborhood and map is called a **chart** and a collection of a whole system of charts such that every point of M is included is known as an **atlas**.
The main problem now will be to define a tangent space. Since we do not have any external space anymore, where can we define the tangent space? We need a definition in terms of internal quantities:

The **Tangent Space** TM_p *of an m-dimensional manifold in point p is defined as*

$$TM_p = \{(\varphi, \xi) | \varphi \quad \textit{is a chart of } M \textit{ in } p, \xi \in \mathbb{R}^n\} / \sim$$

where $(\varphi, \xi) \sim (\psi, \eta) \leftrightarrow d(\psi \circ \varphi^{-1})_{\varphi(p)}(\xi) = \eta$. *So the tangent space is the set of all classes* $TM_p = [\varphi, \xi]$.

It can be shown, that an equivalent definition is given in terms of the coordinate functions $x^1, ..., x^n$: we define a **derivation** in p as a linear map $X : C^\infty(M) \to \mathbb{R}$ with the property

$$X(f \circ g) = X(f)g(p) + f(p)X(g) \quad \forall f, g \in C^\infty(M).$$

Next, we define the **canonical derivation** as

$$\frac{\partial}{\partial x^i}|_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)).$$

It follows that the tangential space TM_p can be proved to be the vector space of all the derivations at the point p, spanned by the set of canonical derivations:

$$[\varphi, \xi] = X = \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial x^j}|_p \in TM_p.$$

This definition is motivated by the analogy with submanifolds in an external space, like $\mathbb{R}^n$, where the tangent space is easily seen to be the space formed

by the directional derivatives to all the curves on M at the point p.
The **cotangent space** TM_p^* *is defined as the dual to* TM_p*:*

$$TM_p^* = \{\lambda : TM_p \to \mathbb{R}, linear\}$$

The basis of TM_p^* is obviously the dual basis to TM_p,

$$dx_p^i\Big(\frac{\partial}{\partial x^j}|_p\Big) = \frac{\partial x^i}{\partial x^j}(p) = \delta_j^i.$$

The **Tangent Bundle** TM *and the* **Cotangent Bundle** TM^* *are defined as the set of all points of the manifold together with their tangent (cotangent) space in that point:*

$$TM = \bigcup_{p\in M} \{p\} \times TM_p \qquad TM^* = \bigcup_{p\in M} \{p\} \times TM_p^*$$

The Tangent Bundle and the cotangent Bundle are examples of what are called vector bundles over M, namely fiber bundles, whose fibers are real (or complex) vector spaces.
A bundle is a manifold which is the product of a fiber and a manifold, but the delicacy lies in how this product is taken: *a smooth manifold E is a* **vector bundle** *over a smooth manifold M if there exists a smooth projection* $\pi : E \to M$ *with the properties:*

1. $\forall p \in M$ *has* $E_p = \pi^{-1}(p)$ *the structure of a real vector space, and*
2. $\forall p \in M$ *it exists an open neighborhood* U *of* p *in* M *and a smooth diffeomorphism* $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$*, so that* $\psi|E_q$ *a vector space isomorphism of* E_q *into* $\{q\} \times \mathbb{R}^n$*.*

In our case this definition sustains that the tangential space in every point has the structure of a real vector space, and helps us visualizing the fiber structure of TM, being TM_p the fiber in every point, isomorph to $\mathbb{R}^n$. (e vera questa cosa?)
Finally we define a **smooth section** of E as any smooth function s with the property

$$\pi \circ s = id_M, \text{which means } s(p) \in E_p = \pi^{-1}\ \forall p \in M$$

Such a function relates to each point of the manifold a set of points in the fiber.

After having defined some features of the structure of a real manifold, we shall say something about another kind of objects, which can be thought as the ones acting on this structures: differential forms and tensors.

We use the notation $C^\infty(E)$ for the vector space of smooth sections of E. *Elements of* $C^\infty(TM)$ *are called* **vector fields** *and elements of* $C^\infty(TM^*)$ *are called* $\mathbf{1-}$**forms** *(or differential forms).* We can expand this definition to the $k-th$ exterior power of the cotangent bundle $\Lambda^k TM^*$ (which is also a real vector bundle over M) and define *a* $\mathbf{k-}$**form** *as a smooth section of* $\Lambda^k TM^*$*.*
k-forms also form a vector space, written $C^\infty(\Lambda^k TM^*)$.
A nowhere vanishing n-form on an n-dimensional manifold M is called a **volume form**. It can be shown that, if M is orientate, then there exists only one volume form on M.

The natural operation between forms is the so called **exterior product** (or wedge product) $\wedge$ which, if α is a k-form and β an l-form, takes α and β into a $(k+l)$-form $\alpha \wedge \beta$:

$$(\alpha \wedge \beta) = \frac{(k+l)!}{k!l!} Alt(\alpha \otimes \beta) \quad \in \Lambda^{k+l}(TM^*)$$

Where $Alt(\cdot)$ is the linear projection on the space of antisymmetric tensors (see later):

$$Alt(\alpha)(v_1, ..., v_k) = \frac{1}{k!} \sum_\sigma sgn(\sigma) \alpha(v_{\sigma(1)}, ..., v_{\sigma_(s)}$$

Where σ is a permutation of the indexes $[1, ..., k]$ and v_i are elements of the tangent space. (?vero?)

So, a k-form can be written uniquely using the wedge product, being $dx^1, ..., dx^n$ the basis of TM^*:

$$\alpha_k = \alpha_{\mu_1,...,\mu_k} dx^{\mu_1} \wedge ... \wedge dx^{\mu_k} \quad (\mu_i \in 1, ..., n)$$

for some smooth functions $\alpha_{\mu_1,...,\mu_k} : U \to \mathbb{R}$. Here we have used the Einstein summation convention and summed over repeated indices. An $(n+1)$-form in n dimensions always vanishes, because of his anti-symmetry A zero form is just a smooth real function on a subset of M and a k-form, being $dx^1, ..., dx^n$ the basis of TM^*, can be written uniquely using the wedge product:

$$\alpha_k = \alpha_{\mu_1,...,\mu_k} dx^{\mu_1} \wedge ... \wedge dx^{\mu_k} \quad (\mu_i \in 1, ..., n)$$

for some smooth functions $\alpha_{\mu_1,\dots,\mu_k} : U \rightarrow \mathbb{R}$. Here we have used the Einstein summation convention and summed over repeated indices. An $(n+1)$-form in n dimensions always vanishes, because of his anti-symmetry. And a zero-form is just a smooth function on a subset of M.

The natural derivative operator on forms is the **exterior derivative** (or curl), which maps a k-form into a $k+1$-form, in index notation:

$$(d\alpha)_{a_1 \dots a_{k+1}} = \frac{\partial}{\partial x^{[a_1}} \alpha_{a_2 \dots ak+1]}$$

Here denoted [..] antisymmetrizaton over the enclosed indices.
So, for example, a 2-form α with coordinates α_{ij}:

$$d\alpha = \frac{\partial \alpha_j k}{\partial x^i} dx^i \wedge dx^j \wedge dx^k,$$

Here the antisymmetrization comes from the definition of the wedge product. An important property of the exterior derivation is that for any k-form α

$$d(d\alpha) = d^2 \alpha = 0$$

which follows from the commutability of partial derivatives.
Studying the exterior derivation on forms we can group k-forms into **closed** ones, for which $d\alpha = 0$, and **exact** ones, for which there exists some globally defined $(k-1)$-form β with $\alpha = d\beta$. An exact form is obviously closed (from $d^2 = 0$) and an n-form on an n-dimensional manifold is closed, as well. One can show that on $\mathbb{R}^n$ every closed form is exact (since the definition of manifold states that every point has a neighborhood injectively mapped to $\mathbb{R}^n$). Hence, locally on M every closed form is exact. But this is not in general true globally. The obstruction to doing so is the k-th **De Rahm cohomology** $H^k(M)$:

$$H^k(M) = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}}$$

which is the set of all closed k-forms, where two forms α and α' are considered equivalent if $\alpha - \alpha'$ is exact. The elements of $H^k(M)$ are equivalence classes called **cohomology classes** $[\alpha]$ and they build a real vector space, which dimension is called the **k-th Betti number** $b_k(M) = dim(H^k(M))$. The simplest example for a simple connected manifold is $H^0(M)$, the space of constant functions, with, of course, $b_0 = 0$.
Since on $\mathbb{R}^n$ every closed form can be written as an exact form, and the De

Rahm cohomology has just one cohomology class, we could say that the De Rahm cohomology is in some sense a measure of how non-trivial the topology of M is.

Now, taking the tensor product instead of the exterior product of the vector bundles TM and TM^*, we obtain the bundles of tensors on M. A **(k,l)-Tensor** T on M is a smooth section of a bundle $\bigotimes^k TM \otimes \bigotimes^l TM^*$. In analogy to k-forms we can write tensors in coordinates, too, remembering that $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$ forms a basis of TM:

$$T = \sum_{1 \leq a_i \leq n,\ 1 \leq i \leq k,\ 1 \leq b_j \leq n,\ 1 \leq j \leq l} T^{a_1 \ldots a_k}_{b_1 \ldots b_l} \frac{\partial}{\partial x^{a_1}} \otimes ... \otimes \frac{\partial}{\partial x^{a_k}} \otimes dx^{b_1} \otimes ... \otimes dx^{b_l}$$

Now $\Lambda^k TM^*$ is a subbundle of $\bigotimes^k TM^*$, so k-forms are tensors on M, as well, tensors $\alpha_{a_1 \ldots a_k}$ with k covariant indices that are antisymmetric (they change sign when two of its arguments are exchanged).

2.2 Curvature and Holonomy Groups

In this section we will introduce the two most important concepts of this chapter: the curvature and the holonomy group of a real manifold. These will be of great importance when coming to Calabi-Yau manifolds, where a simple restriction upon the holonomy group or upon the curvature, forces a manifold to be Calabi-Yau. As a natural path in defining curvature and holonomy group, we will also meet connections, a sort of directional derivative on manifolds, which, together with parallel transport, provides a tool for comparing tangent vectors at different points.

Tangent vectors in TM_p and tangent vectors in $TM_{p'}$ cannot be added together as if they would if the manifold was embedded in some external space, since they don't lay in the same space anymore. A first link between tangent spaces at different points is given by so-called connections. *If M is a manifold and TM his tangent bundle, then a* **connection** ∇^{TM} *on TM is a bilinear map $\nabla^{TM} : C^\infty(TM) \to C^\infty(TM \otimes TM^*)$ satisfying*

$$1) \quad \nabla^{TM}(\alpha Y) = \alpha \nabla^{TM} Y + Y \otimes d\alpha, \tag{2.1}$$

whenever $Y \in C^\infty(TM)$ is a vector field and $\alpha : M \to \mathbb{R}$ is a smooth function on M.

If ∇^{TM} is such a connection, and $X \in C^\infty(TM)$, then we write

$$\nabla_X^{TM}(Y) = X \cdot \nabla^{TM} Y \in C^\infty(TM) \tag{2.2}$$

where the dot " $\cdot$ " contracts together the TM factor in X and the TM^* factor in $\nabla^{TM} Y$ (since TM^* is the space of linear functions $TM \to \mathbb{R}$, we can "contract" them by applying them to functions in TM, and obtaining a real number).
Then, for a smooth function on M $\beta : M \to \mathbb{R}$ we have

$$2) \quad \nabla_{\beta X}^{TM} Y = \beta \nabla_X^{TM} Y. \tag{2.3}$$

Note that for any $Y \in C^\infty(TM)$ fix, $\nabla Y : C^\infty(TM) \to C^\infty(TM)$ defines a $(1,1)$-tensor field, that is, $(\nabla_X Y)_p$ depends only on X at the point p and, vice versa, for any $X \in C^\infty(TM)$ fix, it depends only upon Y in a neighborhood of p. This motivates the name connection, since it creates a link between tangent space in different points (Come lo crea il link?).

We are now ready to introduce the curvature of a connection. This will be of particular interest when dealing with Riemannian manifolds, where, thank to the uniqueness of the Levi-Civita connection, the curvature will become an intrinsic property of the metric.
The idea is that, *for every connection* ∇^{TM}, *there exists a unique tensor* $R : C^\infty(TM) \times C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ *satisfying*

$$R(X,Y)Z = \nabla_X^{TM} \nabla_Y^{TM} Z - \nabla_Y^{TM} \nabla_X^{TM} Z - \nabla_{[X,Y]}^{TM} Z \tag{2.4}$$

for all $X, Y, Z \in C^\infty(TM)$, *where* $[X,Y]^a = X^b \frac{\partial Y^a}{\partial x^b} - Y^a \frac{\partial X^a}{\partial x^b}$ *is the lie bracket of* X, Y. *This tensor is called the* **curvature of** $\mathbf{\nabla^{TM}}$.
The reason why it is written $R(X,Y)Z$ is that for fixed X and Y, we can see $R(X,Y)$ as an endomorphism of the tangent space.
In a local chart, denote by R^l_{kij} the l-th component of $R(\partial/\partial x^i, \partial/\partial x^j)\partial/\partial x^k$, these are the components of the curvature tensor.
Let's now have a look to the practical meaning of the curvature. Taking X and Y to be two different element of the basis, $X = v_i = \frac{\partial}{\partial x^i}$ and $Y = v_j = \frac{\partial}{\partial x^j}$, then $[v_i, v_j] = 0$. Now, with Z a vector field, we can interpret $\nabla_{v_i}^{TM} Z$ as a kind of partial derivative $\partial Z/\partial x^i$ of Z (perche'????). From the definition of the curvature we get

$$R(X,Y)Z = \frac{\partial^2 Z}{\partial x^i \partial x^j} - \frac{\partial^2 Z}{\partial x^j \partial x^i} \tag{2.5}$$

which means that the curvature may be interpreted as a sort of measure of how much partial derivatives fail to commute in TM.
Another tensor which can be defined for connections is the **torsion of $\nabla^{\mathbf{TM}}$**, a unique tensor $T : C^{\infty}(TM) \times C^{\infty}(TM) \to C^{\infty}(TM)$ with the property

$$T(X,Y) = \nabla_X^{TM} Y - \nabla_y^{TM} X - [X,Y]. \tag{2.6}$$

A connection with $T = 0$ is called $\mathbf{torsion-free}$.

We have been talking about connections as a link between tangent spaces in different points $p \in M$. This becomes touchable with the introduction of parallel transport, a linear map effectively transporting vectors from the tangent space in one point $p \in M$ to the tangent space in another point $q \in M$, where they can be compared and summed to each other. Let's now put together some definitions yielding to parallel transport.
A vector field Y is called **parallel**, *if $\nabla_X^{TM} Y = 0$ for all $X \in C^{\infty}(TM)$.*
In the same way a vector field Y^c along a curve $c : I \subset \mathbb{R} \to M$ ("along" means that $Y^c \in TM_{c(t)} \forall t \in I$) is called **parallel along this curve** *if $\nabla_{\dot{c}}^{TM} Y^c = 0$.*
Now, from the existence and uniqueness of the solution of differential equations it follows:

Proposition 1 *Let be $c : [a,b] \to M$ a curve in M, then, for each $v \in TM_{c(a)}$ it exists only one parallel vector field Y_v^c along c with $Y_v^c(a) = v$.*

Thus, we can define a map, bringing vectors v from the tangent space in $p = c(a)$ into $v' = Y_v^c(b)$, in the tangent space to $q = c(b)$. This map

$$P^c : TM_{c(a)} \to TM_{c(b)}, \quad P^c(v) = Y_v^c(b) \tag{2.7}$$

is called **parallel transport**. It may help to note that for submanifolds in $\mathbb{R}^n$, a vector is said to be parallel transported along the curve c if its change as it moves along c is orthogonal to the tangent plane, i.e. it is not rotated in the tangent plane.
A curve which parallel transports the vector tangent to the curve itself is called a **geodesic**.
Now we might ask what happens if we parallel transport a vector around a closed curve. The answer is that it is generally found to be rotated when it returns to its initial location. The group of all such rotations is called **holonomy group**

$$Hol_p(\nabla^{TM}) = \{p^c : TM_p \to TM_p | c : [a,b] \to M \text{ piecewise-smooth path}$$

$$\text{with } c(a) = c(b) = p\} \subset GL(TM_p). \tag{2.8}$$

Before talking about the relations between holonomy group and curvature, we shall list some of his properties.

Property 1 *$Hol_p(\nabla^{TM})$ is a Lie subgroup of $GL(TM_p)$.*

Proof This follows from $P^{\gamma\delta} = P^{\gamma} \circ P\delta$ and $P^{\gamma^{-1}} = (P^{\gamma})^{-1}$, where γ and δ are two piecewise-smooth closed curves in p, $\gamma\delta$ is the path which goes through γ and then δ and γ^{-1} is the curve γ with inverse parameterization. This proves $Hol_p(\nabla^{TM})$ being a subgroup, which is why we call it group.$\diamondsuit$

Property 2 *$Hol_p(\nabla^{TM})$ is independent of the basepoint $p \in M$ and $Hol_p(\nabla^{TM})$ as a subgroup H of $GL(n, \mathbb{R})$ defined up to conjugation in $GL(n, \mathbb{R})$.*

Proof This is somewhat more abstract to imagine: suppose M be connected (this will be the case through all the paper), then, for each $p, q \in M$ we can find a piecewise-smooth path $\gamma : [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$, so that $P^{\gamma} : TM_p \to TM_q$. Then $Hol_p(\nabla^{TM})$ and $Hol_q(\nabla^{TM})$ satisfy

$$P^{\gamma} Hol_p(\nabla^{TM}) P^{-\gamma} = Hol_q(\nabla^{TM}) \tag{2.9}$$

Now, if TM_p has fiber $\mathbb{R}^n$ (recall the definition of vector bundle), then any identification $TM \cong \mathbb{R}^n$ induces an isomorphism $GL(TM_p) \cong GL(n, \mathbb{R})$. Then we might regard $Hol_p(\nabla^{TM})$ *as a subgroup H of $GL(n, \mathbb{R})$ defined up to conjugation in $GL(n, \mathbb{R})$*. This follows because, if we had chosen another identification $TM_p \cong \mathbb{R}^n$ we would have gotten another subgroup aHa^{-1} of $GL(n, \mathbb{R})$ for some $a \in GL(n, \mathbb{R})$. Then from (2.9) (remember that now P^{γ} lyes in $Gl(n, \mathbb{R})$, as well) and from what we have just shown, that the holonomy group be defined up to conjugation, we conclude that $Hol_p(\nabla^{TM})$ is independent of the basepoint p, and we can drop this index.$\diamondsuit$

Property 3 *The Holonomy group of a simply connected manifold is also simply connected.*

Next we shall investigate closer the link between holonomy group and curvature. If we parallel transport a vector v around an infinitesimal loop on the manifold: the change in the vector δv is proportional to curvature tensor times the vector self, times the area of the loop. This gives us a hint of a link between this two concepts, but let's look closer to it.
There is a fundamental relationship between the holonomy group (its Lie algebra, actually) and the curvature of a connection: the holonomy group

both constraints the curvature and is determined by it. The next proposition will illustrate such constraints.
We first define the **holonomy algebra** $\mathfrak{hol}_p(\nabla^{TM})$ to be the Lie algebra of $Hol_p^0(\nabla^{TM})$, a vector subspace of $End(TM)$ (end(TM) coincide con qualcos'altro?E poi perche in END?? cos'e' end??), where

$$Hol_p^0(\nabla^{TM}) = \{P^\gamma | \gamma \; null - homotopic \; loop \; based \; in \; p\} \tag{2.10}$$

is the so called **restricted holonomy group**, which is a subgroup of $Hol_p(\nabla^{TM})$, namely the connected component containing the identity. A null-homotopic loop based in p is a piecewise-smooth closed curve which can be continuously deformed to the constant loop in p. Note that, since we are only dealing with simply connected manifolds, also the holonomy group will be simply connected, and therefore the Lie algebras of $Hol^0(\nabla^{TM})$ and $Hol(\nabla^{TM})$ coincide.
With this new concepts in mind, we can come to

Proposition 2 *Let M be a manifold, TM its tangent bundle, and ∇^{TM} a connection on TM. Then for each $p \in M$ the curvature R_p of ∇^{TM} lies in $\mathfrak{hol}_p(\nabla^{TM}) \otimes \Lambda^2 TM_p^*$.*

(beweis?) This is how the holonomy group of a connection constraints its curvature: through its Lie algebra.

The last property of the holonomy group we would like to treat in this chapter, is his influence on so-called constant tensors. But before introducing constant tensors we shall say something else about connections. Up to now we have only seen connections acting on vector fields, but in the very definition of connection (2.2) there is nothing stopping the argument Y from being a general tensor:

Proposition 3 *Let M be a manifold. Then a connection ∇^{TM} on TM induces connections on all the vector bundles of tensors on M. such as $\bigotimes^k TM \otimes \bigotimes^l TM^*$. All of these induced connections on tensors will also be written ∇^{TM}.*

We are now ready to define a **constant** tensor S as a tensor satisfying $\nabla^{TM} S = 0$.
The next theorem states the influence of the holonomy group upon such tensors: constant tensors are determine entirely by the holonomy group.

Theorem 4 *Let M be a manifold, and ∇^{TM} a connection on TM. Let be $p \in M$ a fixed point on M and $H = Hol_p(\nabla^{TM})$. Then H acts naturally on*

the tensor powers $\bigotimes^k TM \otimes \bigotimes^l TM^*$.
Suppose $S \in C^\infty(\bigotimes^k TM \otimes \bigotimes^l TM^*)$ *is a constant tensor, then* $S|_p$ *is invariant under the action of* H *on* $\bigotimes^k TM \otimes \bigotimes^l TM^*$.
Conversely, if $S|_p \in \bigotimes^k TM \otimes \bigotimes^l TM^*$ *is invariant under* H*, then* $S|_p$ *may be extended to a unique constant tensor* $S \in C^\infty(\bigotimes^k TM \otimes \bigotimes^l TM^*)$.

Proof The first part follows simply from the extension of Proposition 3 of ∇ to the whole vector bundle $C^\infty(\bigotimes^k TM \otimes \bigotimes^l TM^*)$.
For the second part, we shall consider that a constant tensor is invariant under parallel transport, in the sense that $P^\gamma(S|_p) = S|_q$. This is because a constant tensor doesn't need a parallel tensor field (as in proposition 1, a parallel vector field) to be transported from one point to another, since it his itself parallel, i.e. $\nabla S = 0$ for each and every curve. So, for any path γ from p to q we have $P^\gamma(S|_p) = S|_q$. Now, taking p and q to be the same point we have $P^\gamma(S|_p) = S|_p$ for any closed curve γ in p. This is like stating that $S|_p$ is fixed by the action of the holonomy group H on TM_p.$\diamondsuit$
This theorem will help us studying the holonomy group of Riemannian manifolds, in next section.

2.3 Riemannian metrics

In this section we will concentrate on a special kind of manifold, Riemannian manifolds. We shall investigate how, by forcing a metric structure on the manifold, we also put restrictions upon the holonomy group, which, as seen in the last section, is strictly related to the curvature. We will revisit some of the concepts we have just treated, and investigate their new, stronger, properties.

A **Reimannian metric** *g on a manifold M is a $(0,2)$-tensor field with the property that, for all $p \in M$, is $g_p : TM_p \times TM_p \to \mathbb{R}$ a scalar product (that is, symmetric and positive definite). The pair manifold and Riemannian metric (M, g) is called a* **Riemannian manifold**.
Once defined Riemannian manifolds, let's say something about their connections.
We are looking for intrinsic properties of the manifold, so it disturb us to have an holonomy group and a curvature defined for each connection; we would much better like to have just one unique connection, strongly related to the metric of the manifold self, and then define holonomy and curvature on that very connection. The next theorem states the existence and uniqueness of such a preferred connection.

Theorem 5 (Fundamental Theorem of Riemannian Geometry) *Let M be a manifold and g a Riemannian metric on M. Then there exists a unique, torsion-free connection ∇ on TM with $\nabla g = 0$.*

This connection is called the **Levi – Civita connection**.
Proof This theorem follows from the properties of torsion-free and constant tensors and the definition of connection, which, combined together and re-arranged are equivalent to

$$2g(\nabla_u v, w) = u \cdot g(v, w) + v \cdot g(u, w) - w \cdot g(u, v) + g([u, v], w)$$
$$-g([v, w], u) - g([u, w], v) \qquad (2.11)$$

where $[v, w] = \nabla_u v - \nabla_v u$. Now it is easy to show that, for fixed u, v, there is a unique vector field satisfying (2.11) for all $w \in C^\infty(TM)$. This is enough to define ∇ uniquely, and it turns out that ∇ is indeed a torsion-free connection with $\nabla g = 0$.$\diamondsuit$

As seen in the previous section the curvature of a connection is a tensor R^l_{kij}, this will also be true for the Levi-Civita connection. Consider now the 4-covariant tensor $R(X, Y, Z, T) = g[X, R(Z, T)Y]$ with components $R_{lkij} = g_{lm} R^m_{kij}$. Both these tensors are known as the **Riemann curvature** of g and have the following symmetries:

$$R_{ijkl} = -R_{ijlk} = R_{klij} \qquad (2.12)$$

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \qquad (2.13)$$

$$\nabla_m R_{ijkl} + \nabla_k R_{ijlm} + \nabla_l R_{ijmk} = 0 \qquad (2.14)$$

where (2.13) and (2.14) are known as **bianchi identities**.
Now, if (M, g) is a Riemannian manifold with Riemann curvature R^i_{jkl}, then there is only one nonzero tensor obtained by contraction of the curvature tensor. It is called the **Ricci curvature** of g and its components are $R_{ij} = R^k_{ikj}$. From (2.12) it follows that the Ricci tensor is symmetric.

We would now like to investigate the holonomy group of Riemannian manifolds. We have just explored the new symmetries possessed by the curvature of a Riemannian manifold, and we remember the holonomy group being strictly related to the curvature, so we may expect the holonomy group of a Riemannian manifold having stronger properties than holonomy groups on arbitrary vector bundles.
It is natural to define the **holonomy group of g** $Hol_p(g)$ as the holonomy

group of the Levi-Civita connection of g, let's now investigate its properties. First of all remember the definition of the Levi-Civita connection, which included $\nabla g = 0$, i.e. g is a constant tensor. It follows from theorem 4 that the metric tensor g is invariant under $Hol(g)$. This means that an element of $Hol_p(g)$ cannot be a simple linear map in $GL(TM_p)$ but has to preserve scalar product, and we know that the subgroup of $GL(n, \mathbb{R})$ preserving the scalar product is just $O(n)$. We have therefore showed

Proposition 6 *Let M be an n-manifold, and ∇ a torsion-free connection on TM. Then ∇ is the Levi-Civita connection of a Riemannian metric g on M if and only if $Hol(\nabla)$ is conjugate in $GL(n, \mathbb{R})$ to a subgroup of $O(n)$.*

Next we shall explore how the holonomy group of Riemannian manifold constraints its curvature.
Define the **holonomy algebra $\mathfrak{hol}(g)$ of g** to be the holonomy algebra of the Levi-Civita connection. Then, since we have seen that $\mathfrak{hol}(\nabla)$ is the Lie algebra of the connected component containing the unity element, $\mathfrak{hol}(g)$ will be a Lie subalgebra of $\mathfrak{so}(n)$. Now, for every $p \in M$ is $\mathfrak{hol}_p(\nabla)$ a vector subspace of $TM_p \otimes TM_p^*$ (perche??????), which can be identified with $\otimes^2 TM_p^*$, if we use the metric and equate T_b^a with $T_{ab} = g_{ac}T_b^c$. So $\mathfrak{hol}_p(\nabla)$ is now a vector subspace of $\otimes^2 TM_p^*$, which we wrote as $\mathfrak{hol}_p(g)$. It can further be shown (?) that actually lies $\mathfrak{hol}_p(g)$ in the vector space of the twice covariant antisymmetric tensors $\Lambda^2 TM_p^*$. Next, from Proposition 2 we know that the curvature lies in $\mathfrak{hol}_p(\nabla^{TM}) \otimes \Lambda^2 TM_p^*$.
So, from this and the symmetry properties of the curvature (2.12), it follows that the curvature itself lies in the symmetric part of $\mathfrak{hol}_p(g) \otimes \mathfrak{hol}_p(g)$:

Theorem 7 *Let (M, g) be a Riemannian manifold with Riemannian curvature R. Then R lies in the vector subspace $S^2\mathfrak{hol}_p(g)$ in $\Lambda^2 TM_p^* \otimes \Lambda^2 TM_p^*$ at each $p \in M$.*

This explains the restriction on curvature and holonomy group, posed by the introduction of a metric structure.

Finally we would like to add something about the structure group, a concept which might help understanding the role of the metric and could also be useful in comprehend the step taken in next chapter, to complex manifolds.
We recall the definition of vector bundle, in which we had an homeomorphism

$$\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n \tag{2.15}$$

for some neighborhood U_α of a point $p \in M$. Taking another neighborhood U_β of p we obtain another homeomorphism Ψ_β. In overlapping regions $U_\alpha \cap U_\beta$ the Ψ's are related by the transition functions

$$\Psi_\alpha \cdot \Psi_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^n \to U_\alpha \cap U_\beta \times \mathbb{R}^n \tag{2.16}$$

where $\mathbb{R}^n$ is the fiber. For fixed $p \in M$ this homeomorphisms define the **structure group** of the fiber.
The point is that, if these functions are C^k, then the manifold M is said to be C^k, and if they are holomorphic functions, the manifold is said to be complex (we shall return on this point in next chapter).
Let us now clarify the role of the metric, in the context of structure groups. In general the structure group of a real manifold is $GL(n, \mathbb{R})$. The assignment of a Riemannian metric corresponds to an assignment of a scalar product in the tangent space at each point of the manifold and reduces $GL(n, \mathbb{R})$ to simply $O(n)$.
This is a particular example of the general case where $GL(n, \mathbb{R})$ is reduced to a subgroup G, called $\mathbf{G-structure}$. Another example is $G = GL(n/2, \mathbb{C})$, the group of invertible $n/2 \times n/2$ complex matrices. In this case G is said to be an **almost complex structure**. We will reencounter this definition in the next chapter, where a complex manifold will be a real manifold, allowing a complex structure. (Legame Structure Gruop, Holonomy Group?).

2.4 Example of a Real Manifold

Let us now pause and consider an example. As said in the introduction, we are looking for a real manifold, allowing a Riemannian metric, but which can not be made complex (which fits in A but not in B, where A and B are the sets defined in the prelude). One such manifold is the six-dimensional sphere S^6. Let's consider the standard metric of constant curvature (com'e' definita?posso prima definire la metrica e poi calcolarne la curvatura?). For the connection defined by this metric, the holonomy group is $SO(6)$, since there are no subspaces of the tangent space left invariant under parallel transport(cosa vuol dire?, se ho degli spazi invarianti, il gruppo cambia, come?).

Chapter 3

Complex Manifolds

In the path taking us to Calabi-Yau manifolds, the next condition to impose is that the manifold be a complex one. In this short chapter we shall encounter this new objects and discuss the differences between these and the already known real manifolds. We shall start by exposing two different definitions, a more algebraic one and a more differential geometric one, and then by showing their equivalence. We shall then discuss the meaning of this complexification for tensors and forms, and finally we will give an example of such a manifold.

3.1 Definitions

The traditional definition of complex manifold is the one given at the end of last chapter.
A real manifold of complex dimension $2m$ *is said to be a* **complex manifold** *if it has an atlas of complex charts* (U, Ψ)*, such that all the transition functions are holomorphic, as maps from* $\mathbb{C}^m$ *to itself.* A **complex chart** on M is simply the complex version of a chart, namely a pair (U, Ψ) where U is open in M and $\Psi : U \to \mathbb{C}^m$ is a diffeomorphism between U and some open set in $\mathbb{C}^m$.
Note that, given a $2m$-dimensional real manifold, one can always let the coordinates become independent complex variables and obtain a $2m$ complex dimensional manifold. This is not what is meant here, since here we obtain an n complex dimensional manifold. This means that, in passing from $2m$ real to m complex dimensions, one has to create a link between pairs of coordinates, and this cannot be done on every manifold. This link is highlighted by the so called complex structure, which is the center of a

more differential geometric definition of complex manifold.
So, the second definition is based on the concept of complex structure, therefore, before defining complex manifolds in this new way, we shall start by expose some definitions.
An **almost complex structure** J on a $2m$-dimensional real manifold M is a field of automorphisms $J : TM \to TM$ of the tangent bundle satisfying

$$J_p^2 = -I_p \quad \forall p \in M \tag{3.1}$$

where I denotes the identity.
Observe that, the fact that J is an automorphism and has complex eigenvalues, means that a tangent vector multiplied by a complex number must still lie in TM. Thus, the almost complex structure gives each tangent space TM_p the structure of a complex vector space.
An almost complex structure is said to be **integrable** if the so called **Nijenhuis tensor** $N(J)$ associated to J

$$N(v,w) = [v,w]+J[Jv,w]+J[v,Jw]-[Jv,Jw] \qquad \forall v,w \in C^{\infty}(TM) \tag{3.2}$$

equals zero $N(J) = 0$.
This integrability condition implies that the almost complex structure be induced by a (unique) complex structure. This takes us to define a **complex structure** as an almost complex structure satisfying the integrability condition.
It turns out that this integrability condition is precisely what is needed so one can introduce complex coordinates with holomorphic transition functions.
So we will define a **complex manifold** *simply as a real manifold equipped with a complex structure.*

Let's be more precisely about why this definition is valid; a smooth function $f : M \to \mathbb{C}$ is holomorphic if it satisfy the Cauchy-Riemann equations, which can be written using J as $J(df) \equiv i(df)$. It turns out that if $m > 1$ the equations are overdetermined and the Nijenhuis tensor (perche lui?) forbids the existence of large numbers of holomorphic functions on M. But this is exactly what is needed to form holomorphic coordinates in every point. The *Newlander-Nirenberg Theorem* shows that the condition for the existence of a holomorphic atlas (recall the first definition of complex manifold) is the vanishing of the Nijenhuis tensor. This makes the two definitions equivalent to each other.
Note that a complex structure is not a unique feature for a manifold. A real manifold can give rise to different inequivalent complex manifolds, in other

words, it may have more than one complex structure, even a whole space of structures.

3.2 Complexification of the Tangent and Tensor Bundles

In this section we shall clarify the role of the complex structure, we will see that this induces a splitting of the complexified tangent bundle in two subbundles. In order to give a clear exposition for those already familiar with real manifolds, we will try to work only with real objects, almost avoiding the use of complex coordinates.

We have seen in last chapter how the metric reduced the structure group from $GL(n, \mathbf{R})$ to $O(n)$; we encounter even more dramatic consequences, when introducing a complex structure.
It can be shown, that, if a Lie-group G induces a G-structure on a manifold M, then the bundles of tensors and forms decompose into the direct sum of subbundles, corresponding to the irreducible representations of G. Now, thank to the complex structure, the structure group of a complex manifold becomes $GL(m, \mathbb{C})$ ($m = n/2$ is the complex dimension), so the bundle of tensors and forms will split into subbundles, corresponding to the irreducible representations of $GL(m, \mathbb{C})$.(su cosa?)
We shall now investigate what happens under the action of J, explicitly. Let's first complexify the tangent space in each point, i.e. extend it to complex coordinates. What we will get is $TM_p \otimes_{\mathbb{R}} \mathbb{C}$ (perche?? e cos'e' xr?), a complex vector space isomorphic to $\mathbb{C}^{2m}$ (remember that the real tangent space was isomorphic to $\mathbb{R}^{2m}$).
Now, we have seen that $J : TM \to TM$ acts linearly on vector fields. Since $J^2 v = -v$, for all $v \in TM$, then J must have eigenvectors with eigenvalues $\pm i$. This is how J induces a splitting, in its two different eigenspaces: $TM_p \otimes_{\mathbb{R}} \mathbb{C} = TM_p^{(1,0)} \oplus TM_p^{(0,1)}$, where $TM_p^{(1,0)} \cong \mathbb{C}^m$ ($TM_p^{(0,1)} \cong \mathbb{C}^m$) is the eigenspace with eigenvalue $+i$ ($-i$); these are complex conjugate spaces. This splitting extends to the whole tangent bundle and, similarly, it can also be applied to the complexified cotangent bundle.
Every vector field can be written as a sum

$$X = U + \overline{U} \tag{3.3}$$

where

$$U = \frac{1}{2}(X - iJX) \quad \in TM^{(1,0)} \quad \textit{and} \quad \overline{U} = \frac{1}{2}(X + iJX) \quad \in TM^{(0,1)} \tag{3.4}$$

This operation is a projection.
In a similar way, also the complexified tensor bundle is split into subbundles by the complex structure. We will see what happens to the bundle of $\mathbb{C}$-valued k-forms. Here the situation is a little more complicated, since we have to deal with the exterior product of a sum of two bundles. This becomes

$$\Lambda^k TM^* \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{j=0}^{k} \Lambda^{j,k-j} M \tag{3.5}$$

with

$$\Lambda^{p,q} M = \Lambda^p TM^{*(1,0)} \otimes \Lambda^q TM^{*(0,1)}. \tag{3.6}$$

This is the decomposition of the bundle of k-forms on M induced by the complex structure.
A section of $\Lambda^{(p,q)} M$ is called a $(p,q) -$ **form**.
A projection like (3.3) can also be defined with general tensors, simply by substituting the vector field X with the tensor S and by taking the contraction of S and J in (3.4).
If M is an m-dimensional complex manifold, then, in general $\Lambda^{p,0} M$ has fibre with complex dimension $\binom{m}{p}$, which means that for $p = m$, $\Lambda^{m,o}$ has fibre isomorph to $\mathbb{C}$. This is the bundle of complex volume forms on M and is called the **canonical bundle** of M, written K_M.

Finally we will revisit the exterior derivative d in this new context. One might ask on which part of a (p,q)-form will the derivative operator act. The answer is that the exterior derivative $d\varphi$ of any (p,q)-form φ is the sum of a form of type $(p+1,q)$ and a form of type $(p,q+1)$, which we will write $\partial\varphi$ and $\overline{\partial}\varphi$. So we obtain two differential operators acting on complex forms and satisfying

$$d = \partial + \overline{\partial} \tag{3.7}$$

where ∂ is the component of d mapping $C^\infty(\Lambda^{(p,q)} M)$ into $C^\infty(\Lambda^{(p+1,q)} M)$ and $\overline{\partial}$ is the one mapping $C^\infty(\Lambda^{(p,q)} M)$ into $C^\infty(\Lambda^{(p,q+1)} M)$.
From the properties of d it follows:

$$\partial^2 = \overline{\partial}^2 = 0 \quad \text{and} \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{3.8}$$

3.3 Example of a Complex Manifold

Before continuing to the next chapter, we shall pause a moment on an example. S^6, the real manifold threaten in last chapter doesn't allow a complex structure, since it doesn't allow irreducible eigenspaces on his tangent bundle (e giusta sta cazzata?)
The example we have chosen for this chapter is also a six-dimensional real manifold, namely $S^3 \times S^3$.
If we consider the standard metric on each S^3, then its connection has holonomy group $SO(3) \times SO(3)$ (cos'e x? e poi devo dimostrare che questo e' un sottoinsieme di...di cosa?). We shall see in the next chapter why this example accepts a complex structure (questo potrei gia dirlo qui, PErche???), but doesn't satisfy the Kähler conditions.

Holonomy di una complex manifold???

Chapter 4

Kähler Manifolds

In this chapter we will treat the next class of manifold of interest to us: the ones with a Kähler metric. These are the "nicest" class of metrics on a complex manifold, in that the Riemannian structure is compatible with the complex structure in a natural way. Like every restriction upon the metric we have treated so far, also this one will have consequences for the holonomy group. In fact the holonomy group of a Kähler metric g on a complex m-dimensional manifold, will be a subgroup $Hol(g) \subset U(m)$.
We will start by defining the hermitian form of g, which will be the basepoint in defining Kähler metrics and than Kähler manifolds. Thereafter we will study the holonomy group and, of course, the curvature of these manifolds. Calabi-Yau manifolds will then spontaneously follow by forcing a last restriction upon the curvature of Kähler metrics, i.e. by making them Ricci-flat.

4.1 Definitions

A **hermitian metric** on a complex manifold (M, J) is a Riemannian metric g on M satisfying

$$g(v, w) = g(Jv, Jw) \quad \forall v, w \in C^{\infty}(TM). \tag{4.1}$$

This condition forces a natural compatibility between metric and complex structure.
(4.1) is like asking that the scalar product $g(\cdot, \cdot)$ between any two of the projected vector fields of (3.4) in the same projected space (eigenvalues $+i$ or $-i$) be zero $g(\overline{V}, \overline{U}) = g(V, U) = 0$. So an hermitian metric is one for

which the complex structure divides the tangent bundle in perpendicular subbundles.
For any hermitian metric we can build a 2-form on M, called **hermitian form** ω and defined as

$$\omega(v, w) = g(Jv, w) \quad \forall v, w \in C^{\infty}(TM). \tag{4.2}$$

ω is a $(1,1)$-form in the sense of (3.6).
Note that if we know the hermitian form ω of a metric, we can always reconstruct the metric self with $g(v, w) = \omega(v, Jw)$. This will be of particular interest when studying Calabi-Yau manifolds, where, thank to the so called Calabi conjecture, we will be able to recognize which forms will be the hermitian forms of a Calabi-Yau manifold; from the hermitian form it could then be useful to know how to reconstruct the metric g.

We are now ready to define *a* **Kähler metric** *g on a complex manifold (M, J) as an hermitian metric g with a hermitian form ω satisfying the closure condition $d\omega = 0$. In this case we call ω a* **Kähler form** *and (M, J, g) a* **Kähler manifold**.
Remember when, in the second chapter, we introduced the volume form (which we shall write dV_g), as a unique nowhere vanishing m-form on an m-dimensional manifold. For Kähler manifolds this is just the m-th exterior power of the Kähler form: $\omega^m = m! dV_g$.

Before continue, we shall motivate the ambitious title of "nicest" class of metrics on a complex manifold, given at the beginning of this chapter.

Proposition 8 *Let M be a manifold of dimension $2m$, J an almost complex structure on M, and g an Hermitian metric, with hermitian form ω. Let ∇ be the Levi-Civita connection of g, then the following conditions are equivalent:*

(i) *J is a complex structure and g is Kähler ($d\omega = 0$),*
(ii) $\nabla J = 0$,
(iii) $\nabla \omega = 0$,
(iv) *The holonomy group $Hol(g)$ of g is contained in $U(m) \subset O(2m)$.*

Proof This follows from the definitions of the Kähler form and Hermitian metrics and from the fact that the Nijenhuis-tensor be zero for complex structures, thereafter is just an algebraic computation to show the equivalence of (*i*), (*ii*) and (*iii*). Part (*iv*) is a consequence of (*ii*) and (*iii*), they imply that J and ω be constant tensors on M. From Theorem 4 it follows that they must be invariant under the action of the holonomy group. This restricts the holonomy group to $U(m)$, the subgroup of $O(2m)$ preserving these two tensors.$\diamondsuit$

Now, since the Levi-Civita connection ∇ is the connection of the metric g, the fact that J be constant with respect to g ($\nabla J = 0$) means that the Riemannian structure is somehow compatible with the complex structure. This gives Kähler manifolds stronger properties than normal manifolds.

We said in the introduction that Kähler metrics form an infinite set; next we shall discuss how, given a Kähler metric on M, it is possible to build an infinite number of other Kähler metrics upon the same manifold.
To achieve this, we shall translate the problem of finding Kähler metrics into the problem of finding Kähler forms: we have seen that for every Kähler metric there is a Kähler form, in the other way, every $(1,1)$-form ω is the Kähler form of a Kähler metric if and only if ω is **positive**, which means

$$\omega(v, Jv) > 0 \quad \forall v \in C^\infty(TM),\ v \neq 0. \tag{4.3}$$

Now, it can be shown that,

Proposition 9 (The $\partial\overline{\partial}$-Lemma) *Let $\phi : M \to \mathbb{R}$ be a smooth function on a complex manifold M, then $i\partial\overline{\partial}\phi$ is a closed $(1,1)$-form on M. Conversely, every closed real $(1,1)$-form can be written* locally *as $i\partial\overline{\partial}\phi$, for a smooth function $\phi : M \to \mathbb{R}$.*

This applies also to the Kähler form ω, which can always be described locally by a so called **Kähler potential** ϕ: $\omega = i\partial\overline{\partial}\phi$.
Note that we have emphasized the word locally. This is because it is in general not possible to find a global Kähler potential on a compact manifold M, here is why. From the definition of Kähler manifold it follows that the Kähler form be closed ($d\omega = 0$). So it's an element of the de Rham cohomology $[\omega] \in H^2(M)$, which, for the Kähler form, is called **Kähler class**. Now, it turns out that the volume form on M, which has to be non-zero, depends only on this cohomology class $[\omega]$. So $[\omega]$ must be different from zero, and this is not possible if $\omega = i\partial\overline{\partial}\phi$, since $i\partial\overline{\partial}\phi$ is exact, i.e. $[i\partial\overline{\partial}\phi] = 0$.
So, we cannot write the Kähler form globally with a Kähler potential, but what we can show is that, if two Kähler metrics g, g' have Kähler forms in the same Kḧaler class, then they differ by a Kähler potential:

Theorem 10 *Let (M, J) be a compact complex manifold, and let g, g' be two Kähler metrics on M, with Kähler forms ω, ω' lying in the same Kähler class $[\omega] = [\omega'] \in H^2(M)$. Then there exists a smooth real function $\phi : M \to \mathbb{R}$ on M, such that $\omega' = \omega + i\partial\overline{\partial}\phi$. The function ϕ is unique up to addition with a constant.*

Proof This follows from the fact that if $[\omega] = [\omega']$, than $\omega - \omega'$ is exact. Then, from the definition of exact forms on real manifolds it can be shown that every exact $(1,1)$-form on a complex manifold can be written as $i\partial\overline{\partial}\phi$. Thus, we can also write $\omega - \omega' = i\partial\overline{\partial}\phi$ which is what we wanted to show.$\diamondsuit$

Theorem 9 furnishes a tool for, knowing a Kähler form on M, building others such forms, simply by adding a Kähler potential. As shown before, this is equivalent to the problem of finding Kähler metrics on a given manifold; so we have shown that, if there exist one such metric than we can easily build an infinite-dimensional family of these.

Finally, we would like to say something about the Riemannian curvature on Kähler manifolds.
Like in the case of real and complex manifold, here also, a restriction upon the holonomy gives a restriction upon the curvature. We have seen in Proposition 7, part (iv) that the holonomy group of Kähler manifolds is limited to $U(m)$ for an m-dimensional complex manifold. The main consequence of such a restriction is that the curvature tensor will posses more symmetries. This additional symmetries will be inherited by the Ricci tensor R_{ij}, defined in chapter 2 as the only non-vanishing contraction of the Riemannian curvature. In the following we shall investigate the consequences of this new symmetries.
We have seen at the beginning of this chapter how to create the Kähler form ω, from the metric tensor g as $\omega(v, w) = g(Jv, w)$ for any two vector fields v, w. Two bilinear forms, related by such a relationship are said to be **associated** with each other. We have also seen that the metric tensor can always been reconstructed from his associated Kähler form.
So, let's now try to build such an associated form for the Ricci tensor, as well. This is known as the **Ricci form** ρ, and in index notation is defined as

$$\rho_{ij} = J_i^l R_{lj}. \tag{4.4}$$

si puo definire senza indici?)
So ρ is a real (1,1)-form and, as for the Kähler form, we can always recover the Ricci curvature from ρ with $R_{ij} = \rho_{il} J_j^l$.

Now, the important role of those additional symmetries we have seen before is that they make the Ricci form be a closed form $d\rho = 0$. Therefore the Ricci form defines a cohomology class $[\rho] \in H^2(M)$. This cohomology class turns out to be independent of the Kähler metric you start with, but depends only on the complex structure of M.
So we define the **first Chern class** c_1:

$$[\rho] = 2\pi c_1(M). \tag{4.5}$$

(perche? 2pi???) Note that we decided to define the first Chern in this way, from the Ricci form, but the first Chern class is actually an intrinsic object which can be defined independently from ρ and is related to it by (4.5).

Now, it is obvious that if $c_1 \neq 0$, then there cannot exist a Ricci flat metric (recall the reconstruction from ρ to the Ricci tensor). The converse, namely that there exists a Ricci flat metric for $c_1 = 0$, is far from obvious. However Calabi has proved the uniqueness and Yau, almost twenty years later, has shown the existence of Ricci flat metrics whenever $c_1 = 0$.
But this is another story...
We will threat Calabi-Yau spaces in next chapter.

4.2 Example of a Kähler Manifold

At the end of last chapter we have seen $S^3 \times S^3$ as an example of complex manifold. However this doesn't admit a Kähler metric. It can be shown that for a manifold to admit a Kähler metric, the even Betti numbers must satisfy $b_{2p} \geq 1$ (perche?, non c'e' un altro modo di dirlo, senza betti?), a condition not satisfied by $S^3 \times S^3$, which has $b_2 = 0$.
Thus we will take a similar example: $S^2 \times S^2 \times S^2$, another 6-dimensional manifold. Here also, the holonomy group is very easy, namely $SO(2) \times SO(2) \times SO(2)$. (come dimostro che e' un sottoinsieme di (U(6)?).

Chapter 5

Calabi-Yau Manifolds

Calabi-Yau spaces are named after two mathematicians, Eugenio Calabi and Shing-Tung Yau which, together, found which form could be the Ricci form of a Kähler metric. This is known as the Calabi conjecture, stated by Calabi in 1954. Calabi also proved the uniqueness of such a metric, but for the whole proof the world had to wait another twenty years, since it wasn't until 1976 that Yau proved its existence.
Although the Calabi conjecture furnishes a tool to build Calabi-Yau manifolds in large numbers, their exclusive properties make them very rare indeed. Even though belonging to a family of ten thousands or more may not seem such a privileged position, it has to be compared with the infinite number of possible shapes.

Unfortunately so far nobody has proved the uniqueness of a definition of Calabi-Yau manifolds; so our first task will be to define such objects in different manners, and then to prove that these are just one and the same. Thereafter it will be the turn of the Calabi conjecture. After having seen this powerful theorem, we shall explore some more applicable corollary before spending some time on its proof. Since our approach has so far been mainly differential geometrical, we will skip the analytical part of the proof, the one which took Yau so long to show.
Finally...

5.1 Equivalent Definitions

There are several ways to define Calabi-Yau manifolds; we encountered seven different definitions in the literature, which at first sight looked all equiva-

lent, but after a closer investigation showed little discrepancies: the manifolds defined in one did not fulfilled all the properties required by other definitions. So we combined these seven to three, stronger definitions.
Each one of the following statements can be taken as the definition of **Calabi-Yau spaces**, and they are all equivalent to each other.

(i) Calabi-Yau manifolds are compact Kähler manifolds of dimension $m \geq 2$ with holonomy $Hol(g) = SU(m)$.

(ii) Calabi-Yau manifolds are compact Ricci-flat Kähler manifolds (M, J, g) of complex dimension m with trivial canonical bundle (or M simply connected).

(iii) Calabi-Yau manifolds are quadruples (M, J, g, Ω) such that (M, J) is a compact complex manifold of dimension m, g a Kähler metric on M with holonomy $Hol(g) = SU(m)$, and Ω a nowhere vanishing constant $(m, 0)$-form on M, called the **holomorphic volume form** and which satisfies

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \overline{\Omega}. \tag{5.1}$$

where ω is th Kähler form of g.

Proof of the equivalence of (i), (ii) and (iii)
(i)↔(iii): Let's first see what happens on the simplest imaginable manifold $\mathbb{C}^m \simeq \mathbb{R}^2 m$: define a metric g a $(1, 1)$-form (?)ω and a $(m, 0)$-form Ω on $\mathbb{C}^m$ by

$$g = |dz_1|^2 + ... + |dz_m|^2,$$
$$\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + ... + dz_m \wedge d\bar{z}_m), \quad \Omega = dz_1 \wedge ... \wedge dz_m. \tag{5.2}$$

Now, it is easy to study this objects, when acting on $\mathbb{C}^m$. By simply applying linear transformation to the coordinates z_i, we can verify that the subgroup of $GL(2m, \mathbb{R})$ which preserves g, ω and Ω is simply $SU(m)$. We can now come back to a general manifold M and apply Theorem 4 to this tensors. We see that every Riemannian manifold (M,g) with holonomy $SU(m)$ admits natural like the ones defined in (5.2), and they must be constant under the Levi-Civita connection. Next, if the forms defined in (5.2) satisfy (5.1) in $\mathbb{C}^m$, which can be easily proved, then they will satisfy (5.1) at each point. Finally, it is obvious that there is only one complex structure J on M for which $\omega_{ij} = J_i^k g_{kj}$. So if we identify ω with a Kähler form, then (M, g, J) is a Kähler manifold with Kähler form ω. Furthermore, Ω is a holomorphic (since formed only with holomorphic functions) $(m, 0)$-form, where the

$(m, 0)$- splitting is consistent with J. With this we have proved that every Riemannian manifold with holonomy $SU(m)$ is a Kähler manifold, admitting a constant, holomorphic volume form.
The other way round, a Kähler manifold equipped with a holomorphic volume form with $\nabla\Omega = 0$ must have $Hol(g) = SU(m)$.$\diamondsuit$
(ii)$\leftrightarrow$ *(iii)* We have threaten in chapter 4 the canonical bundle K_M, the bundle of volume forms on M. From the definition of holomorphic volume form Ω, it follows that Ω is a non-zero section of K_M. Such a form, like the one defined in (5.2) exists if and only if the canonical bundle is isomorph to $M \times \mathbb{C}$ (we than say that K_M is **trivial**). We said also that the first Chern class was a topological property of the manifold; it turns out that if the canonical bundle is trivial, then the first Chern class vanishes. Thus, any Kähler manifold with holonomy $SU(m)$ must have $c_1(M) = 0$ and trivial canonical bundle. This can be considered a fourth definition of Calabi-Yau manifold and its equivalence with definition (ii) will be shown thank to the Calabi conjecture, in Corollary 12.$\diamondsuit$

5.2 The Calabi Conjecture

In this section we will threat the famous Calabi Conjecture. This powerful theorem specifies which closed $(1, 1)$-form can be the Ricci form of a Kähler metric on M. By asking that the metric be Ricci-flat and the manifold be simply-connected (or with trivial canonical bundle), we than have the Ricci form of a Calabi-Yau manifold. Thus, the Calabi conjecture represents a useful tool for building large number of Calabi-Yau spaces, and this is how they got their name.
Note that the fact that we know the Ricci form is not enough to know the metric explicitly. Although from the Ricci form we can reconstruct the Ricci tensor, then we can go no further, since the Ricci tensor is a contraction of the Riemannian curvature, we cannot work it out explicitly.(vero?)

Theorem 11 (The Calabi conjecture) *Let M be a compact, complex manifold and g a Kähler metric on M, with Kähler form ω. Given any real closed $(1, 1)$-form ρ' on M, such that $[\rho'] = 2\pi c_1(M)$, there exists a unique Kähler metric g' on M with Kähler form ω', such that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$ and the such that the Ricci form of g' is ρ'.*

The interesting part of this theorem is that, if we have a vanishing first Chern class $c_1(M) = 0$, then we can choose ρ' to be zero and, as a result

of the Calabi conjecture we will have a Ricci-flat metric g': the metric of a Calabi-Yau space.
So we have proved the uniqueness of Calabi-Yau metrics in every Kähler class:

Corollary 12 *Let (M,J) be a compact complex manifold with $c_1(M) = 0$. Then every Kähler class on M contains a unique Ricci-flat Kähler metric g.*

We are now ready to begin the proof of the Calabi conjecture. As explained before, we shall concentrate on the differential geometrical aspects of the proof, the ones proved by Calabi, and skip its algebraic difficulties, the ones solved by Yau.
The main idea of the proof is to translate the problem of finding existence and uniqueness of a metric with given properties, to the problem of showing existence and uniqueness of a second-order nonlinear elliptic partial differential equation. So we will show how to rewrite this equivalent version of the Calabi conjecture.

Proof of the Calabi conjecture
Restating the problem. Let (M, J) be a compact complex manifold, g a Kähler metric with Kähler form ω on M, and let ρ be the Ricci form of g. Next, choose a real, closed $(1,1)$-form ρ' on M such that $[\rho'] = 2\pi c_1(M)$. What we are looking for is a metric g', with Kähler form ω' in the same Kähler class of ω, $[\omega'] = [\omega]$, such that g' has Ricci form ρ'. Finding g' would be the proof of the Calabi conjecture.
In this first part of the proof, we shall restate the problem of finding the metric g', to the problem of finding the solution of a partial differential equation (p.d.e.) for a real function ϕ on M.

We will start by considering the following Lemma, which can be proven by working out explicitly the hermitian form and the Ricci form of a given metric g.

Lemma 13 *Let ω be the Kähler form of a complex m-dimensional manifold (M,J) and ω_0 the form induced (?) by the standard hermitian form on $\mathbb{C}^m$, given by*

$$\omega_0^m = \frac{(-1)^{m(m-1)/2} i^m m!}{2^m} dz_1 \wedge ... \wedge dz_m \wedge d\overline{z_1} \wedge ... \wedge d\overline{z_m} \tag{5.3}$$

for holomorphic coordinates $(z_1, ..., z_m)$ on an open set U of M. Let define a smooth function $F : U \to [0, \infty]$ by

$$\omega^m = F\omega_0^m. \tag{5.4}$$

Then the Ricci form is given by $\rho = -i\partial\overline{\partial}(log(F))$, on U.

This Lemma can be further generalized: let ω and ω' be two Kähler forms on M, with Ricci forms ρ, ρ', and let define a function $\tilde{F} : M \to \mathbb{R}$ by $(\omega')^m = \tilde{F}\omega^m$. Then we can write $(\omega')^m = \tilde{F}F\omega_0$, where F and ω_0 are defined in the Lemma. Applying Lemma 13 to this new form, and using the properties of the logarithm, leads

$$\rho' = -i\partial\overline{\partial}log(\tilde{F}F) = -i\partial\overline{\partial}log(\tilde{F}) - i\partial\overline{\partial}log(F) = \rho - i\partial\overline{\partial}log(\tilde{F}).$$

Taking $\tilde{F} = e^f$ for a smooth function $f : M \to \mathbb{R}$, we have

$$\rho' - \rho = -i\partial\overline{\partial}f. \tag{5.5}$$

Ok, let's now come back to our problem. We have those two forms ρ and ρ' and we know they satisfy $[\rho'] = [\rho] = 2\pi c_1(M)$ which means that $\rho - \rho'$ is exact. Thus, from the $\partial\overline{\partial}$-Lemma, we get an expression like (5.5), with f unique up to addition of a constant $f + c$. We can further get rid of this constant by invoking the volume form: recall that the volume form of a Kähler metric g with Kähler form ω, was defined from the Kähler form as $m!dV_g = \omega^m$. We have said before that this volume form depends only of the Kähler class of ω and since we have $[\omega] = [\omega']$, we can write $\int_M (\omega')^m = \int_M \omega^m$. Furthermore we have seen that this volume form is unique and can be thought as a sort of volume of our manifold $vol_g(M)$. This fixes the constant c:

$$e^c \int_M e^f dV_g = \int_M dV_g = vol_g(M).$$

Without lack of generality, we can assume that this constant vanishes, $c = 0$. Magically, all these calculations brought us to formulate the Calabi conjecture in a second, equivalent way:

Theorem 14 (The Calabi conjecture, second version) *Let (M,J) be a compact complex manifold with Kähler metric g and Kähler form ω. Let $f : M \to \mathbb{R}$ be smooth, then there exists a unique Kähler metric g' on M with Kähler form ω', such that $[\omega'] = [\omega]$ and $(\omega')^m = e^f\omega^m$.*

Does this looks easier? It doesn't but it is easier. This new statement is about the existence of a metric with a prescribed volume form $(\omega')^m$, the first statement was about the existence of a metric with prescribed Ricci form. This new volume form should also be positive and should have the same total volume as the other volume form ω^m:

$$m!dV_{g'} = (\omega')^m >^! 0 \quad m!dV_g = \omega^m \quad \Rightarrow \quad \int_M (\omega')^m = \int_M \omega^m. \tag{5.6}$$

This means that the second version of the Calabi-conjecture states that now, there is only one metric g' in the same Kähler class as g, whose volume form $dV_{g'}$ fulfills (5.6).
The simplification lies in the fact that the Ricci curvature depends also on the second derivatives of the metric tensor g', while the volume form depends only on g'.
But this is not yet enough to be solved with analytical methods; we would like to have something similar, but where the thing to find is not a form but a function. This is easily done by applying the $\partial\overline{\partial}$-Lemma to ω and ω'. Recall that these two forms lie in the same class, which, after this lemma, means that they differ by a Kähler potential, a smooth real function ϕ on M, such that:

$$\omega' = \omega - i\partial\overline{\partial}\phi. \tag{5.7}$$

The $\partial\overline{\partial}$-Lemma also states that this function be unique up to addition of a constant, so we will do no harm if we just ask to fix this constant by $\int_M \phi dV_g = 0$.
Therefore we can now write ω' as in (5.7), and this translates the problem of finding a volume form ω' fulfilling (5.6), to the problem of finding a smooth, real function fulfilling:

Theorem 15 (The Calabi conjecture, third version) *Let (M, J) be a compact complex manifold, with metric g and Kähler form ω. Let f be a smooth real function on M, such that $\int_M e^f \omega^m = \int_M \omega^m$. Then there exists a smooth real function ϕ on M, satisfying*

(i) $\omega + i\partial\overline{\partial}\phi$ *is the Kähler form of some Kähler metric g'*

(ii) $\int_M \phi dV_g = 0$

(iii) $(\omega + i\partial\overline{\partial}\phi)^m = e^f \omega^m$ *on M.*

Note that, although (i) sounds quite a hard task, we have seen in chapter four that it is enough for $\omega + i\partial\overline{\partial}\phi$ to be a positive $(1,1)$ form, in order to

be the Kähler form of some Kähler metric.
Writing part (iii) explicitly, using holomorphic coordinates and the metric tensor, leads to a non-linear, elliptic, second-order partial differential equation in ϕ, of a kind known as Monge-Ampère equation.
So, the Calabi conjecture is now just an analytical problem: solving this Monge-Ampère equation and finding ϕ. Proving that such an equation has a solution for every suitable function f is a very hard task, solved by Yau using the so called continuity method. Unfortunately this lies beyond our possibilities here.
Proving uniqueness What we can do, is prove the uniqueness of such a solution, which, after our opinion, may be even of more interest for us, in the context of this paper. This was done by Calabi and it will take us just a few steps.
Assume that there are two different metrics ω_1 and ω_2, which means (after the proof of the existence of a solution, for the third version of the Calabi conjecture) that we have two different smooth real functions ϕ_1 and ϕ_2, such that

$$\omega_1 = \omega + i\partial\overline{\partial}\phi_1, \quad \text{and} \quad \omega_2 = \omega + i\partial\overline{\partial}\phi_2 \tag{5.8}$$

Since we are interested in showing that there is no difference, between two solutions, we can, without loss of generality, assume that $\phi_2 = 0$. So, calling $\phi_1 = 0$ and putting $\omega_1^m = \omega_2^m$ (since the two volume forms must be equal, recall the second version of the conjecture), we have

$$\begin{aligned} 0 &= \omega_2^m - \omega_1^m \\ &= \omega^n - (\omega + i\partial\overline{\partial}\phi)^n \end{aligned} \tag{5.9}$$

using the binomial formula leads

$$\begin{aligned} 0 &= (\omega - (\omega + i\partial\overline{\partial}\phi) \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega_1 + ... + \omega \wedge \omega_1^{n-2} + \omega_1^{n-1}) \\ &= -i\partial\overline{\partial}\phi \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega_1 + ... + \omega \wedge \omega_1^{n-2} + \omega_1^{n-1}). \end{aligned} \tag{5.10}$$

We then multiply this by ϕ and integrate over the whole manifold,

$$0 = -i \int_M \phi\partial\overline{\partial}\phi \wedge (\omega^{n-1} + ... + \omega_1^{n-1}) \tag{5.11}$$

which, after partial integration, gives

$$0 = i \int_M \partial\phi \wedge \overline{\partial}\phi \wedge (\omega^{n-1} + ... + \omega_1^{n-1}). \tag{5.12}$$

Now, recall that the condition for a $(1,1)$-form ω to be Kähler, is that it has to be positive: $\omega, \omega_1 \geq 0$. So, with $V = \int_M \omega^m$, we get

$$\frac{1}{V}\partial\phi \wedge \overline{\partial}\phi \wedge (\omega^{n-1} \underbrace{+ ... + \omega_1^{n-1}}_{\geq 0}) \geq \frac{1}{V}\partial\phi \wedge \overline{\partial}\phi \wedge \omega^{n-1} \tag{5.13}$$

Now, using Hodge theory, it can be shown that for a general form α, we can write

$$\alpha \wedge \omega^{m-1} = (\alpha, (m-1)!\omega)dV_g \tag{5.14}$$

where the term in brackets is the scalar product of α and ω, defined using the metric tensor. Setting $\alpha = \partial\phi \wedge \overline{\partial}\phi$, and solving the scalar product explicitly in coordinates we find $(\partial\phi \wedge \overline{\partial}\phi, \omega) = \frac{1}{2}|\nabla\phi|^2$. Then, recall that $m!dV_g = \omega^m$. Put all this into (5.14) and solve

$$0 \geq \frac{1}{V}\int_M \partial\phi \wedge \overline{\partial}\phi \wedge \omega^{n-1} = \frac{1}{2nV}\int_M |\nabla\phi|^2\omega^m. \tag{5.15}$$

From this we deduce that $\nabla\phi = 0$ which means that ϕ is constant. A constant ϕ inserted in (5.8), is equivalent to $\omega_1 = \omega_2$. This shows the uniqueness of a solution of the Calabi conjecture, and also finishes our proof.

$\diamondsuit$

5.3 Example of Calabi-Yau Manifold

This time we have chosen the six dimensional torus T^6.

5.4 Calabi-Yau Spaces and string theory

String theory is a new branch of particle theoretical physics born in 1968 after an experimental observation: a number of experimental data could be explained if the particles where assumed to be little vibrating strings.
This idea caught quickly the eye of many theoretical physicists, which liked the idea that all the particles and the force carriers could have the same origin: vibrating strings, which vibrational patterns give rise to different particles.
This theory then encapsulated all the principles of mathematical beauty, and, what was even more surprising, allowed a unification of all the forces of nature. It isn't just a descriptive theory, like the standard model, which at that time was believed to describe completely nature's properties; string

theory goes further, it also explains why things are the way they are.
Unfortunately, the equations governing string theory are so far impossible to solve be interpreted.
One curios feature that emerges from these equations is that they require that the space-time we live in be a manifold M of dimension 10, 11 or 26. In supersymmetric string theory, a popular branch of string theory, this value is 10. The equations of string theory then impose that this manifold M be the product $M = \mathbb{R}^4 \times X$ of the well known 4-dimensional Minkowsky space-time and a six dimensional manifold X. It turns out that this manifold X has the shape of a Calabi-Yau 3-fold.
The reason why we don't see any extra dimension is that they are believed to be curled up in a radius of order $10^{-33}cm$, the so called Plank length. So when we sweep our hand through the air, we circumnavigate these spaces again and again, in every point of our four-dimensional space time, and, being the Plank length so small, there is no room for an extended object like your hand to fit in and, at the end, you will return at the starting point.

Bibliography

[1] T. Aubin, *Nonlinear alalysis on manifolds. Monge-Ampère equatons*, Springer-Verlag, 1982

[2] A.L. Besse, *Einstein Manifolds*, Springer-Verlag, New York, 1987

[3] B. Greene, *The Elegant Univers*, Vintage, 2000

[4] G.T. Horowitz, *What is a Calabi-Yau Space?*, Lucio??

[5] D.D. Joyce, *Compact Manifolds with Special Holonomy*, OUP, Oxford, 2000

[6] D.D. Joyce, *Lectures on Calabi-Yau and special Lagrangian geometry*, arXiv:math.DG/0108088v2 9Nov2001

[7] W. Kühnel, *Differential Geometrie*, Vieweg-verlag, 1999

[8] U. Lang, *Lecture notes on differential geometry*, ETHZ, 2001-2002

[9] G.C Segrè, *Superstrings and Four-Dimensional Physics*, arXiv??

[10] G. Tian, *Canonical metrics in Kähler geometry*, Birkhäuser, 2000