6

EIGENVALUES AND EIGENVECTORS

INTRODUCTION TO EIGENVALUES ■ 6.1

Linear equations $A\boldsymbol{x} = \boldsymbol{b}$ come from steady state problems. Eigenvalues have their greatest importance in *dynamic problems*. The solution of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ is changing with time—growing or decaying or oscillating. We can't find it by elimination. This chapter enters a new part of linear algebra. All matrices in this chapter are square.

A good model comes from the powers $A, A^2, A^3, \ldots$ of a matrix. Suppose you need the hundredth power A^{100}. The starting matrix A becomes unrecognizable after a few steps:

$$\underset{A}{\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}} \quad \underset{A^2}{\begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix}} \quad \underset{A^3}{\begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix}} \quad \cdots \quad \underset{A^{100}}{\begin{bmatrix} .6000 & .6000 \\ .4000 & .4000 \end{bmatrix}}$$

A^{100} was found by using the *eigenvalues* of A, not by multiplying 100 matrices. Those eigenvalues are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A. ***Certain exceptional vectors $\boldsymbol{x}$ are in the same direction as $A\boldsymbol{x}$. Those are the "eigenvectors"***. Multiply an eigenvector by A, and the vector $A\boldsymbol{x}$ is a number λ times the original $\boldsymbol{x}$.

The basic equation is $A\boldsymbol{x} = \lambda\boldsymbol{x}$. The number λ is the ***"eigenvalue"***. It tells whether the special vector $\boldsymbol{x}$ is stretched or shrunk or reversed or left unchanged—when it is multiplied by A. We may find $\lambda = 2$ or $\frac{1}{2}$ or -1 or 1. The eigenvalue λ could be zero! Then $A\boldsymbol{x} = 0\boldsymbol{x}$ means that this eigenvector $\boldsymbol{x}$ is in the nullspace.

If A is the identity matrix, every vector has $A\boldsymbol{x} = \boldsymbol{x}$. All vectors are eigenvectors. The eigenvalue (the number lambda) is $\lambda = 1$. This is unusual to say the least. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues. This section teaches how to compute the $\boldsymbol{x}$'s and λ's. It can come early in the course because we only need the determinant of a 2 by 2 matrix.

For the matrix A in our model above, here are eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Multiplying those vectors by A gives $\boldsymbol{x}_1$ and $\frac{1}{2}\boldsymbol{x}_2$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$:

$$\boldsymbol{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix} = \boldsymbol{x}_1 \quad (A\boldsymbol{x} = \boldsymbol{x} \text{ means that } \lambda_1 = 1)$$

$$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \tfrac{1}{2}\boldsymbol{x}_2 \text{ so } \lambda_2 = \tfrac{1}{2}).$$

If we again multiply $\boldsymbol{x}_1$ by A, we still get $\boldsymbol{x}_1$. Every power of A will give $A^n\boldsymbol{x}_1 = \boldsymbol{x}_1$. Multiplying $\boldsymbol{x}_2$ by A gave $\frac{1}{2}\boldsymbol{x}_2$, and if we multiply again we get $(\frac{1}{2})^2\boldsymbol{x}_2$. ***When A is squared, the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ stay the same***. The λ's are now 1^2 and $(\frac{1}{2})^2$. ***The eigenvalues are squared!*** This pattern keeps going, because the eigenvectors stay in their own directions (Figure 6.1) and never get mixed. The eigenvectors of A^{100} are the same $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The eigenvalues of A^{100} are $1^{100} = 1$ and $(\frac{1}{2})^{100} =$ very small number.

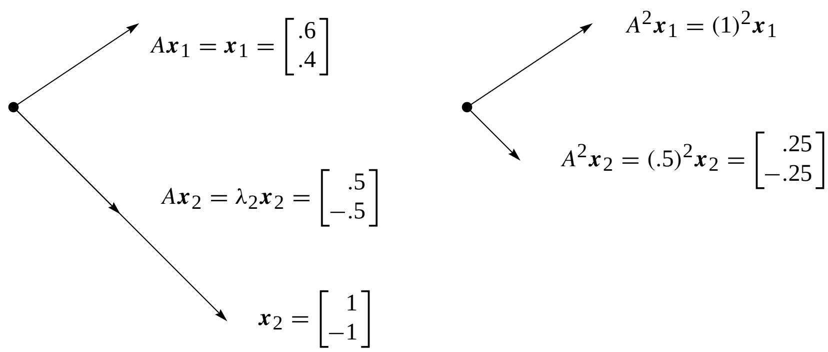


Figure 6.1 The eigenvectors keep their directions. A^2 has eigenvalues 1^2 and $(.5)^2$.

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\boldsymbol{x}_1 + (.2)\boldsymbol{x}_2$:

$$\begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is} \quad \boldsymbol{x}_1 + (.2)\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .2 \\ -.2 \end{bmatrix}. \tag{1}$$

Multiplying by A gives the first column of A^2. Do it separately for $\boldsymbol{x}_1$ and $(.2)\boldsymbol{x}_2$. Of course $A\boldsymbol{x}_1 = \boldsymbol{x}_1$. And A multiplies $\boldsymbol{x}_2$ by its eigenvalue $\frac{1}{2}$:

$$A\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix} \quad \text{is} \quad \boldsymbol{x}_1 + \frac{1}{2}(.2)\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .1 \\ -.1 \end{bmatrix}.$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A. We didn't need these eigenvectors to find A^2. But it is the good way to do 99 multiplications.

At every step $\boldsymbol{x}_1$ is unchanged and $\boldsymbol{x}_2$ is multiplied by $(\frac{1}{2})$, so we have $(\frac{1}{2})^{99}$:

$$A^{99}\begin{bmatrix}.8\\.2\end{bmatrix} \quad \text{is really} \quad \boldsymbol{x}_1 + (.2)(\frac{1}{2})^{99}\boldsymbol{x}_2 = \begin{bmatrix}.6\\.4\end{bmatrix} + \begin{bmatrix}\text{very}\\\text{small}\\\text{vector}\end{bmatrix}.$$

This is the first column of A^{100}. The number we originally wrote as .6000 was not exact. We left out $(.2)(\frac{1}{2})^{99}$ which wouldn't show up for 30 decimal places.

The eigenvector $\boldsymbol{x}_1$ is a "steady state" that doesn't change (because $\lambda_1 = 1$). The eigenvector $\boldsymbol{x}_2$ is a "decaying mode" that virtually disappears (because $\lambda_2 = .5$). The higher the power of A, the closer its columns approach the steady state.

We mention that this particular A is a ***Markov matrix***. Its entries are positive and every column adds to 1. Those facts guarantee that the largest eigenvalue is $\lambda = 1$ (as we found). Its eigenvector $\boldsymbol{x}_1 = (.6, .4)$ is the *steady state*—which all columns of A^k will approach. Section 8.3 shows how Markov matrices appear in applications.

For projections we can spot the steady state ($\lambda = 1$) and the nullspace ($\lambda = 0$).

Example 1 **The projection matrix** $P = \begin{bmatrix}.5 & .5\\.5 & .5\end{bmatrix}$ has eigenvalues 1 and 0.

Its eigenvectors are $\boldsymbol{x}_1 = (1, 1)$ and $\boldsymbol{x}_2 = (1, -1)$. For those vectors, $P\boldsymbol{x}_1 = \boldsymbol{x}_1$ (steady state) and $P\boldsymbol{x}_2 = \mathbf{0}$ (nullspace). This example illustrates three things that we mention now:

1. Each column of P adds to 1, so $\lambda = 1$ is an eigenvalue.

2. P is singular, so $\lambda = 0$ is an eigenvalue.

3. P is symmetric, so its eigenvectors $(1, 1)$ and $(1, -1)$ are perpendicular.

The only possible eigenvalues of a projection matrix are 0 and 1. The eigenvectors for $\lambda = 0$ (which means $P\boldsymbol{x} = 0\boldsymbol{x}$) fill up the nullspace. The eigenvectors for $\lambda = 1$ (which means $P\boldsymbol{x} = \boldsymbol{x}$) fill up the column space. The nullspace is projected to zero. The column space projects onto itself.

An in-between vector like $\boldsymbol{v} = (3, 1)$ partly disappears and partly stays:

$$\boldsymbol{v} = \begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix} \quad \text{projects onto} \quad P\boldsymbol{v} = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix}.$$

The projection keeps the column space part of $\boldsymbol{v}$ and destroys the nullspace part. To emphasize: *Special properties of a matrix lead to special eigenvalues and eigenvectors.* That is a major theme of this chapter (it is captured in a table at the very end).

Projections have $\lambda = 0$ and 1. Permutations have all $|\lambda| = 1$. The next matrix R (a reflection and at the same time a permutation) is also special.

Example 2 **The reflection matrix $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 1 and -1.**
The eigenvector $(1, 1)$ is unchanged by R. The second eigenvector is $(1, -1)$—its signs are reversed by R. A matrix with no negative entries can still have a negative eigenvalue! The eigenvectors for R are the same as for P, because $R = 2P - I$:

$$R = 2P - I \qquad \text{or} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2}$$

Here is the point. If $P\boldsymbol{x} = \lambda\boldsymbol{x}$ then $2P\boldsymbol{x} = 2\lambda\boldsymbol{x}$. The eigenvalues are doubled when the matrix is doubled. Now subtract $I\boldsymbol{x} = \boldsymbol{x}$. The result is $(2P - I)\boldsymbol{x} = (2\lambda - 1)\boldsymbol{x}$. ***When a matrix is shifted by I, each λ is shifted by*** 1. No change in eigenvectors.

Figure 6.2 Projections have eigenvalues 1 and 0. Reflections have $\lambda = 1$ and -1. A typical $\boldsymbol{x}$ changes direction, but not the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

The eigenvalues are related exactly as the matrices are related:

$$R = 2P - I \qquad \text{so the eigenvalues of } R \text{ are} \qquad 2(1) - 1 = 1 \text{ and } 2(0) - 1 = -1.$$

The eigenvalues of R^2 are λ^2. In this case $R^2 = I$. Check $(1)^2 = 1$ and $(-1)^2 = 1$.

The Equation for the Eigenvalues

In small examples we found λ's and $\boldsymbol{x}$'s by trial and error. Now we use determinants and linear algebra. *This is the key calculation in the chapter*—***to solve*** $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

First move $\lambda\boldsymbol{x}$ to the left side. Write the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ as $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$. The matrix $A - \lambda I$ times the eigenvector $\boldsymbol{x}$ is the zero vector. ***The eigenvectors make up the nullspace of*** $A - \lambda I$! When we know an eigenvalue λ, we find an eigenvector by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

Eigenvalues first. If $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ has a nonzero solution, $A - \lambda I$ is not invertible. ***The determinant of $A - \lambda I$ must be zero.*** This is how to recognize an eigenvalue λ:

6A The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular:

$$\det(A - \lambda I) = 0. \tag{3}$$

This "characteristic equation" involves only λ, not $\boldsymbol{x}$. When A is n by n, $\det(A-\lambda I)=0$ is an equation of degree n. Then A has n eigenvalues and each λ leads to $\boldsymbol{x}$:

$$\textbf{For each } \lambda \textbf{ solve} \quad (A-\lambda I)\boldsymbol{x}=\mathbf{0} \quad \text{or} \quad A\boldsymbol{x}=\lambda\boldsymbol{x} \quad \textbf{to find an eigenvector } \boldsymbol{x}. \tag{4}$$

Example 3 $A=\begin{bmatrix}1&2\\2&4\end{bmatrix}$ is already singular (zero determinant). Find its λ's and $\boldsymbol{x}$'s.

When A is singular, $\lambda=0$ is one of the eigenvalues. The equation $A\boldsymbol{x}=0\boldsymbol{x}$ has solutions. They are the eigenvectors for $\lambda=0$. But here is the way to find *all* λ's and $\boldsymbol{x}$'s! Always subtract λI from A:

$$\textbf{\textit{Subtract } } \lambda \textbf{\textit{ from the diagonal to find}} \quad A-\lambda I=\begin{bmatrix}1-\lambda & 2\\ 2 & 4-\lambda\end{bmatrix}.$$

Take the determinant "$ad-bc$" of this 2 by 2 matrix. From $1-\lambda$ times $4-\lambda$, the "ad" part is $\lambda^2-5\lambda+4$. The "bc" part, not containing λ, is 2 times 2.

$$\det\begin{bmatrix}1-\lambda & 2\\ 2 & 4-\lambda\end{bmatrix}=(1-\lambda)(4-\lambda)-(2)(2)=\lambda^2-5\lambda. \tag{5}$$

Set this determinant $\lambda^2-5\lambda$ to zero. One solution is $\lambda=0$ (as expected, since A is singular). Factoring into λ times $\lambda-5$, the other root is $\lambda=5$:

$$\det(A-\lambda I)=\lambda^2-5\lambda=0 \quad \text{yields the eigenvalues} \quad \lambda_1=0 \quad \text{and} \quad \lambda_2=5.$$

Now find the eigenvectors. Solve $(A-\lambda I)\boldsymbol{x}=\mathbf{0}$ separately for $\lambda_1=0$ and $\lambda_2=5$:

$$(A-0I)\boldsymbol{x}=\begin{bmatrix}1&2\\2&4\end{bmatrix}\begin{bmatrix}y\\z\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix}y\\z\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix} \text{ for } \lambda_1=0$$

$$(A-5I)\boldsymbol{x}=\begin{bmatrix}-4&2\\2&-1\end{bmatrix}\begin{bmatrix}y\\z\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix}y\\z\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix} \text{ for } \lambda_2=5.$$

The matrices $A-0I$ and $A-5I$ are singular (because 0 and 5 are eigenvalues). The eigenvectors $(2,-1)$ and $(1,2)$ are in the nullspaces: $(A-\lambda I)\boldsymbol{x}=\mathbf{0}$ is $A\boldsymbol{x}=\lambda\boldsymbol{x}$.

We need to emphasize: *There is nothing exceptional about* $\lambda=0$. Like every other number, zero might be an eigenvalue and it might not. If A is singular, it is. The eigenvectors fill the nullspace: $A\boldsymbol{x}=0\boldsymbol{x}=\mathbf{0}$. If A is invertible, zero is not an eigenvalue. We shift A by a multiple of I to *make it singular*. In the example, the shifted matrix $A-5I$ was singular and 5 was the other eigenvalue.

Summary To solve the eigenvalue problem for an n by n matrix, follow these steps:

1. ***Compute the determinant of*** $A-\lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.
2. ***Find the roots of this polynomial***, by solving $\det(A-\lambda I)=0$. The n roots are the n eigenvalues of A. They make $A-\lambda I$ singular.
3. For each eigenvalue λ, ***solve*** $(A-\lambda I)\boldsymbol{x}=\mathbf{0}$ ***to find an eigenvector*** $\boldsymbol{x}$.

A note on quick computations, when A is 2 by 2. The determinant of $A - \lambda I$ is a quadratic (starting with λ^2). >From factoring or the quadratic formula, we find its two roots (the eigenvalues). Then the eigenvectors come immediately from $A - \lambda I$. This matrix is singular, so both rows are multiples of a vector (a, b). *The eigenvector is any multiple of* $(b, -a)$. The example had $\lambda = 0$ and $\lambda = 5$:

$$\lambda = 0: \text{ rows of } A - 0I \text{ in the direction } (1, 2); \text{ eigenvector in the direction } (2, -1)$$
$$\lambda = 5: \text{ rows of } A - 5I \text{ in the direction } (-4, 2); \text{ eigenvector in the direction } (2, 4).$$

Previously we wrote that last eigenvector as $(1, 2)$. Both $(1, 2)$ and $(2, 4)$ are correct. There is a whole *line of eigenvectors*—any nonzero multiple of $\boldsymbol{x}$ is as good as $\boldsymbol{x}$. MATLAB 's **eig**(A) divides by the length, to make the eigenvector into a unit vector.

We end with a warning. Some 2 by 2 matrices have only *one* line of eigenvectors. This can only happen when two eigenvalues are equal. (On the other hand $A = I$ has equal eigenvalues and plenty of eigenvectors.) Similarly some n by n matrices don't have n independent eigenvectors. Without n eigenvectors, we don't have a basis. We can't write every $\boldsymbol{v}$ as a combination of eigenvectors. In the language of the next section, we can't diagonalize a matrix without n independent eigenvectors.

Good News, Bad News

Bad news first: If you add a row of A to another row, or exchange rows, the eigenvalues usually change. *Elimination does not preserve the* λ's. The triangular U has *its* eigenvalues sitting along the diagonal—they are the pivots. But they are not the eigenvalues of A! Eigenvalues are changed when row 1 is added to row 2:

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{has } \lambda = 0 \text{ and } \lambda = 1; \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{has } \lambda = 0 \text{ and } \lambda = 2.$$

Good news second: The *product* λ_1 *times* λ_2 *and the sum* $\lambda_1 + \lambda_2$ *can be found quickly from the matrix*. For this A, the product is 0 times 2. That agrees with the determinant (which is 0). The sum of eigenvalues is $0 + 2$. That agrees with the sum down the main diagonal (which is $1 + 1$). These quick checks always work:

6B ***The product of the n eigenvalues equals the determinant of A.***

6C ***The sum of the n eigenvalues equals the sum of the n diagonal entries of A.*** This sum along the main diagonal is called the ***trace*** of A:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \textbf{\textit{trace}} = a_{11} + a_{22} + \cdots + a_{nn}. \tag{6}$$

Those checks are very useful. They are proved in Problems 16–17 and again in the next section. They don't remove the pain of computing λ's. But when the computation is wrong, they generally tell us so. To compute correct λ's, go back to $\det(A - \lambda I) = 0$.

The determinant test makes the *product* of the λ's equal to the *product* of the pivots (assuming no row exchanges). But the sum of the λ's is not the sum of the pivots—as the example showed. The individual λ's have almost nothing to do with the individual pivots. In this new part of linear algebra, the key equation is really *nonlinear*: λ multiplies $\boldsymbol{x}$.

Imaginary Eigenvalues

One more bit of news (not too terrible). The eigenvalues might not be real numbers.

Example 4 ***The* 90° *rotation* $Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *has no real eigenvectors or eigenvalues*.** *No vector $Q\boldsymbol{x}$ stays in the same direction as $\boldsymbol{x}$* (except the zero vector which is useless). There cannot be an eigenvector, unless we go to ***imaginary numbers***. Which we do.

To see how i can help, look at Q^2 which is $-I$. If Q is rotation through 90°, then Q^2 is rotation through 180°. Its eigenvalues are -1 and -1. (Certainly $-I\boldsymbol{x} = -1\boldsymbol{x}$.) Squaring Q is supposed to square its eigenvalues λ, so we must have $\lambda^2 = -1$. *The eigenvalues of the* 90° *rotation matrix Q are $+i$ and $-i$*, because $i^2 = -1$.

Those λ's come as usual from $\det(Q - \lambda I) = 0$. This equation gives $\lambda^2 + 1 = 0$. Its roots are $\lambda_1 = i$ and $\lambda_2 = -i$. They add to zero (which is the trace of Q). The product is $(i)(-i) = 1$ (which is the determinant).

We meet the imaginary number i also in the eigenvectors of Q:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = i\begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = -i\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Somehow these complex vectors $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (i, 1)$ keep their direction as they are rotated. Don't ask me how. This example makes the all-important point that real matrices can easily have complex eigenvalues. The particular eigenvalues i and $-i$ also illustrate two special properties of Q:

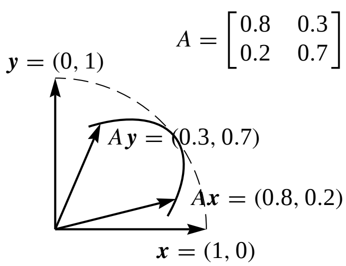
1. Q is an orthogonal matrix so the absolute value of each λ is $|\lambda| = 1$.

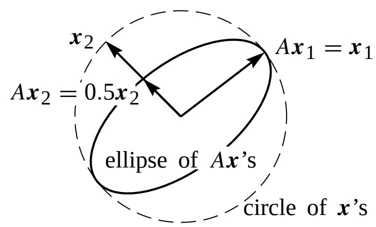
2. Q is a skew-symmetric matrix so each λ is pure imaginary.

A symmetric matrix $(A^{\mathrm{T}} = A)$ can be compared to a real number. A skew-symmetric matrix $(A^{\mathrm{T}} = -A)$ can be compared to an imaginary number. An orthogonal matrix $(A^{\mathrm{T}}A = I)$ can be compared to a complex number with $|\lambda| = 1$. For the eigenvalues those are more than analogies—they are theorems to be proved in Section 6.4. The eigenvectors for all these special matrices are perpendicular. Somehow $(i, 1)$ and $(1, i)$ are perpendicular (in Chapter 10).

Eigshow

There is a MATLAB demo (just type **eigshow**), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector $\boldsymbol{x} = (1, 0)$. *The mouse makes this vector move around the unit circle*. At the same time the screen shows $A\boldsymbol{x}$, in color and also moving. Possibly $A\boldsymbol{x}$ is ahead of $\boldsymbol{x}$. Possibly $A\boldsymbol{x}$ is behind $\boldsymbol{x}$. *Sometimes $A\boldsymbol{x}$ is parallel to $\boldsymbol{x}$*. At that parallel moment, $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (twice in the second figure).





The eigenvalue λ is the length of $A\boldsymbol{x}$, when the unit eigenvector $\boldsymbol{x}$ is parallel. The built-in choices for A illustrate three possibilities:

1. There are *no real eigenvectors*. *$A\boldsymbol{x}$ stays behind or ahead of $\boldsymbol{x}$*. This means the eigenvalues and eigenvectors are complex, as they are for the rotation Q.

2. There is only *one* line of eigenvectors (unusual). The moving directions $A\boldsymbol{x}$ and $\boldsymbol{x}$ meet but don't cross. This happens for the last 2 by 2 matrix below.

3. There are eigenvectors in *two* independent directions. This is typical! $A\boldsymbol{x}$ crosses $\boldsymbol{x}$ at the first eigenvector $\boldsymbol{x}_1$, and it crosses back at the second eigenvector $\boldsymbol{x}_2$.

Suppose A is singular (rank one). Its column space is a line. The vector $A\boldsymbol{x}$ has to stay on that line while $\boldsymbol{x}$ circles around. One eigenvector $\boldsymbol{x}$ is along the line. Another eigenvector appears when $A\boldsymbol{x}_2 = \mathbf{0}$. Zero is an eigenvalue of a singular matrix.

You can mentally follow $\boldsymbol{x}$ and $A\boldsymbol{x}$ for these six matrices. How many eigenvectors and where? When does $A\boldsymbol{x}$ go clockwise, instead of counterclockwise with $\boldsymbol{x}$?

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

■ REVIEW OF THE KEY IDEAS ■

1. $Ax = \lambda x$ says that x keeps the same direction when multiplied by A.
2. $Ax = \lambda x$ also says that $\det(A - \lambda I) = 0$. This determines n eigenvalues.
3. The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1}, with the same eigenvectors.
4. The sum and product of the λ's equal the trace (sum of a_{ii}) and determinant.
5. Special matrices like projections P and rotations Q have special eigenvalues !

■ WORKED EXAMPLES ■

6.1 A Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A + 4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda - 1)(\lambda - 3) = 0$ so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. For the trace, the sum $2 + 2$ agrees with $1 + 3$. The determinant 3 agrees with the product $\lambda_1\lambda_2 = 3$. The eigenvectors come separately by solving $(A - \lambda I)x = \mathbf{0}$ which is $Ax = \lambda x$:

$$\boldsymbol{\lambda = 1}: \quad (A - I)x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{\lambda = 3}: \quad (A - 3I)x = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A + 4I$ keep the *same eigenvectors*. Their eigenvalues are λ^2, λ^{-1}, $\lambda + 4$:

$$A^2 \text{ has } 1^2 = 1 \text{ and } 3^2 = 9 \quad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \quad A + 4I \text{ has } 1 + 4 = 5 \text{ and } 3 + 4 = 7$$

The trace of A^2 is $5 + 5 = 1 + 9 = 10$. The determinant is $25 - 16 = 9$.

Notes for later sections: A has *orthogonal eigenvectors* (Section 6.4 on symmetric matrices). A can be *diagonalized* (Section 6.2). A is *similar* to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.6). A is a *positive definite matrix* (Section 6.5) since $A = A^{\mathrm{T}}$ and the λ's are positive.

6.1 B For which real numbers c does this matrix A have (**a**) two real eigenvalues and eigenvectors (**b**) a repeated eigenvalue with only one eigenvector (**c**) two complex eigenvalues and eigenvectors?

$$A = \begin{bmatrix} 2 & -c \\ -1 & 2 \end{bmatrix} \qquad A^{\mathrm{T}}A = \begin{bmatrix} 5 & -2c-2 \\ -2c-2 & 4+c^2 \end{bmatrix}.$$

What is the determinant of $A^{\mathrm{T}}A$ by the product rule? What is its trace? How do you know that $A^{\mathrm{T}}A$ doesn't have a negative eigenvalue?

Solution The determinant of A is $4-c$. The determinant of $A-\lambda I$ is

$$\det \begin{bmatrix} 2-\lambda & -c \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + (4-c) = 0.$$

The formula for the roots of a quadratic is

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2} = \frac{4 \pm \sqrt{16-16+4c}}{2} = 2 \pm \sqrt{c}.$$

Check the trace (it is 4) and the determinant $(2+\sqrt{c})(2-\sqrt{c}) = 4-c$. The eigenvalues are real and different for $c > 0$. There are two independent eigenvectors $(\sqrt{c}, 1)$ and $(-\sqrt{c}, 1)$. Both roots become $\lambda = 2$ for $c = 0$, and there is only one independent eigenvector $(0, 1)$. Both eigenvalues are complex for $c < 0$ and the eigenvectors $(\sqrt{c}, 1)$ and $(-\sqrt{c}, 1)$ become complex.

The determinant of $A^{\mathrm{T}}A$ is $\det(A^{\mathrm{T}})\det(A) = (4-c)^2$. The trace of $A^{\mathrm{T}}A$ is $5+4+c^2$. If one eigenvalue is negative, the other must be positive to produce this trace $\lambda_1 + \lambda_2 = 9 + c^2$. But then negative times positive would give a negative determinant.

In fact every $A^{\mathrm{T}}A$ has real nonnegative eigenvalues (Section 6.5).

Problem Set 6.1

1 The example at the start of the chapter has

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^{\infty} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

The matrix A^2 is halfway between A and A^{∞}. Explain why $A^2 = \frac{1}{2}(A + A^{\infty})$ from the eigenvalues and eigenvectors of these three matrices.

(a) Show from A how a row exchange can produce different eigenvalues.

(b) Why is a zero eigenvalue *not* changed by the steps of elimination?

2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the _____ eigenvectors as A. Its eigenvalues are _____ by 1.

3 Compute the eigenvalues and eigenvectors of A and A^{-1}:

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the _____ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues _____.

4 Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same _____ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues _____.

5 Find the eigenvalues of A and B and $A+B$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A+B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Eigenvalues of $A+B$ (are equal to)(are not equal to) eigenvalues of A plus eigenvalues of B.

6 Find the eigenvalues of A and B and AB and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Eigenvalues of AB (are equal to)(are not equal to) eigenvalues of A times eigenvalues of B. Eigenvalues of AB (are equal to)(are not equal to) eigenvalues of BA.

7 Elimination produces $A = LU$. The eigenvalues of U are on its diagonal; they are the _____. The eigenvalues of L are on its diagonal; they are all _____. The eigenvalues of A are not the same as _____.

8 (a) If you know $\boldsymbol{x}$ is an eigenvector, the way to find λ is to _____.

(b) If you know λ is an eigenvalue, the way to find $\boldsymbol{x}$ is to _____.

9 What do you do to $A\boldsymbol{x} = \lambda\boldsymbol{x}$, in order to prove (a), (b), and (c)?

(a) λ^2 is an eigenvalue of A^2, as in Problem 4.

(b) λ^{-1} is an eigenvalue of A^{-1}, as in Problem 3.

(c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 2.

10 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Explain why A^{100} is close to A^∞:

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}.$$

11 Here is a strange fact about 2 by 2 matrices with eigenvalues $\lambda_1 \neq \lambda_2$: The columns of $A - \lambda_1 I$ are multiples of the eigenvector $\boldsymbol{x}_2$. Any idea why this should be?

12 Find the eigenvalues and eigenvectors for the projection matrices P and P^{100}:

$$P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

13 From the unit vector $\boldsymbol{u} = \left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$ construct the rank one projection matrix $P = \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$.

(a) Show that $P\boldsymbol{u} = \boldsymbol{u}$. Then $\boldsymbol{u}$ is an eigenvector with $\lambda = 1$.

(b) If $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ show that $P\boldsymbol{v} = \boldsymbol{0}$. Then $\lambda = 0$.

(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

14 Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos\theta \pm i \sin\theta$:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{rotates the } xy \text{ plane by the angle } \theta.$$

Find the eigenvectors of Q by solving $(Q - \lambda I)\boldsymbol{x} = \boldsymbol{0}$. Use $i^2 = -1$.

15 Every permutation matrix leaves $\boldsymbol{x} = (1, 1, \ldots, 1)$ unchanged. Then $\lambda = 1$. Find two more λ's for these permutations:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

16 Prove that the determinant of A equals the product $\lambda_1 \lambda_2 \cdots \lambda_n$. Start with the polynomial $\det(A - \lambda I)$ separated into its n factors. Then set $\lambda =$ _____:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \quad \text{so} \quad \det A = \underline{\qquad}.$$

17 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 - (a+d)\lambda + ad - bc = 0.$$

If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) =$ _____. The quadratic formula gives the eigenvalues $\lambda = (a + d + \sqrt{})/2$ and $\lambda =$ _____. Their sum is _____.

18 If A has $\lambda_1 = 4$ and $\lambda_2 = 5$ then $\det(A - \lambda I) = (\lambda - 4)(\lambda - 5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a + d = 9$ and determinant 20 and $\lambda = 4, 5$.

19 A 3 by 3 matrix B is known to have eigenvalues $0, 1, 2$. This information is enough to find three of these:

(a) the rank of B

(b) the determinant of $B^{\mathrm{T}}B$

(c) the eigenvalues of $B^{\mathrm{T}}B$

(d) the eigenvalues of $(B + I)^{-1}$.

20 Choose the second row of $A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix}$ so that A has eigenvalues 4 and 7.

21 Choose a, b, c, so that $\det(A - \lambda I) = 9\lambda - \lambda^3$. Then the eigenvalues are $-3, 0, 3$:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}.$$

22 ***The eigenvalues of A equal the eigenvalues of A^{T}.*** This is because $\det(A - \lambda I)$ equals $\det(A^{\mathrm{T}} - \lambda I)$. That is true because ____. Show by an example that the eigenvectors of A and A^{T} are *not* the same.

23 Construct any 3 by 3 Markov matrix M: positive entries down each column add to 1. If $\boldsymbol{e} = (1, 1, 1)$ verify that $M^{\mathrm{T}}\boldsymbol{e} = \boldsymbol{e}$. By Problem 22, $\lambda = 1$ is also an eigenvalue of M. Challenge: A 3 by 3 singular Markov matrix with trace $\frac{1}{2}$ has eigenvalues $\lambda =$ ____.

24 Find three 2 by 2 matrices that have $\lambda_1 = \lambda_2 = 0$. The trace is zero and the determinant is zero. The matrix A might not be 0 but check that $A^2 = 0$.

25 This matrix is singular with rank one. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

26 Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Then $A = B$. *Reason*: Any vector $\boldsymbol{x}$ is a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. What is $A\boldsymbol{x}$? What is $B\boldsymbol{x}$?

27 The block B has eigenvalues $1, 2$ and C has eigenvalues $3, 4$ and D has eigenvalues $5, 7$. Find the eigenvalues of the 4 by 4 matrix A:

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

28 Find the rank and the four eigenvalues of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

29 Subtract I from the previous A. Find the λ's and then the determinant:

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

When A (all ones) is 5 by 5, the eigenvalues of A and $B = A - I$ are _____ and _____ .

30 (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

31 When $a + b = c + d$ show that $(1, 1)$ is an eigenvector and find both eigenvalues of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

32 When P exchanges rows 1 and 2 *and* columns 1 and 2, the eigenvalues don't change. Find eigenvectors of A and PAP for $\lambda = 11$:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad \text{and} \quad PAP = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}.$$

33 Suppose A has eigenvalues $0, 3, 5$ with independent eigenvectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Find all solutions.

(c) Show that $A\boldsymbol{x} = \boldsymbol{u}$ has no solution. (If it did then _____ would be in the column space.)

34 *Is there a real* 2 *by* 2 *matrix* (other than I) *with* $A^3 = I$? Its eigenvalues must satisfy $\lambda^3 = I$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What trace and determinant would this give? Construct A.